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STRENGTH AND CREEP OF LAMINATED PLASTICS

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YATSENKO, V. F.

TRANSLATED FROM RUSSIAN

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STRENGTH AND CREEP OF LAMINATED PLASTICS

(PROCHNOST I POLZUCHEST SLOISTYKH PLASTIKOV)

YATSENKO, V. F.

Naukova Dumka, Kiev, 1966

TRANSLATED FROM RUSSIAN

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ANNOTATION

This monograph contains the results of an experimental study on the mechanical characteristics of laminated plastics (plastic glasses and DSP-plastics) and wood subjected to compression, tension and bending in the case of prolonged action of constant load and loading at constant rates. The basic laws of the change of resistibility, strength and deformability with time for laminated plastics and wood under the action of forces have been established. Engineering methods have been worked out for calculating strength and deformability of laminated plastics and wood as elasto-visco-plastic materials with due consideration of the part played by time factor during compression, tension and bending. Rapid methods for determining the characteristics of prolonged strength and deformability of laminated plastics and wood have been proposed.

This book is meant for the workers of scientific and research organizations, higher educational institutes, plant laboratories and project-design bureaus.

FROM THE EDITOR

The industrial application of polymers is impossible without studying their properties and the laws connecting these properties with the time during which the force is applied. The effect of the time factor on the mechanical characteristics is revealed in various ways depending upon the condition and the nature of loading or deformation, and is stipulated by elastic, viscous and plastic properties of the material.

This monograph contains the results of an experimental investigation and the analytical generalization of the laws governing time dependence of strength, resistibility and deformability of laminated plastics and wood subjected to compression, tension and bending. The author has given due consideration to further development of the methods of calculating the bending of beams in limiting conditions, keeping in view the influence of the time factor on the carrying capacity and deformability. The laws of creep in the case of compression and tension have been made the basis for studying the laws of deformability of the fibers of beams subjected to bending. This made it possible to determine the laws governing the deformation of beams subjected to bending, keeping in view the different types of behavior of a material in compressed and stretched zones of the section. The theory of the limiting state of beams subjected to bending, in which the time factor is taken into consideration, gives a basis for their calculation and rational designing, which is important for structural beams with given durability.

F. P. Belyankin,
Academician, AN UkSSR

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INTRODUCTION

Investigations of creep and prolonged strength of materials, which find application in technology, are essential for establishing a scientific basis of the theory of calculation of structures. Long ago, many investigators in the USSR and abroad have pointed out the similarity in the behavior of different materials under the action of a force with the passage of time and have also indicated that the time factor plays a primary role in securing the reliability and longevity of designs.

In all construction and machine-building materials, short-lived or prolonged internal processes take place leading to a change in the mechanical properties of the materials with time under the action of external forces.

Under the action of an external load, materials undergo deformation of one type or the other. The total deformation consists of instantaneous elastic deformation, elasto-viscous deformation developing with time, and plastic deformation, also developing with time.

Instantaneous elastic deformation means deformation which reacts instantaneously to a change in the conditions of application of the load and which increases at a rate equal to the rate of application of the load in the loading process.

If the load acts or changes with time, the formation of instantaneous elastic deformation is followed by viscous deformation developing with time, which, as shown by the experiments of a number of scientists, also belong to the category of elastic deformations.

After the formation of elasto-viscous deformation, plastic deformation starts developing with time. A quantitative re-

lationship between these deformations caused by the action of external forces varies for different materials. For example, in the case of such elasto-visco-plastic materials as laminated plastics and wood, instantaneous deformations attain a maximum value in short-duration tests. Elasto-viscous and plastic deformations are comparatively small in magnitude, if we do not consider the case of compression of wood across the fibers in which a comparatively large plastic deformation appears due to the form of the cells directed along the wood fibers.

The laws of formation and development of deformations and strength with time of elasto-visco-plastic materials under the action of external forces may be explained from a consideration of their structures. These materials consist of two main components: the skeleton having elastic and plastic properties, and the filler, possessing viscous properties. These properties of the structural elements mainly determine the change in strength and deformability of material with time under the action of forces. In the case of loads below a certain magnitude and acting over a long period, the process of deformation with time breaks up into two periods. Instantaneous elastic deformation is formed in the first period and in the second period, elasto-viscous deformation takes place which steadily decreases with time and approaches a definite value in the course of a given time interval. The rate of development of elasto-viscous deformations during an infinitely long period of action of a constant load tends to zero.

In the case of loads which are higher than a certain magnitude, the process of development of deformation breaks up into three periods: in the first and the second periods, elastic deformation is formed and develops with time, whereas in the third period the process of development of plastic deformations takes place, under a constant load, at a constant rate.

The consideration of the time factor makes it possible to relate the laws of formation and development of deformation and resistibility of elasto-visco-plastic bodies with different types and conditions of loading and deformation, to explain the influence of such factors as rate of loading or deformation on resistibility and deformability, and also to take into account cyclic loading, appearance of the hysteresis phenomenon in the process of loading and unloading, relationship between the creep of material under constant load and the relaxation of stresses caused by a given deformation, etc.

A generalization of the laws obtained in the case of prolonged action of a constant load permits us to derive analytical expressions for determining the values of mechanical characteristics of materials by means of short-duration tests carried out at a given rate of loading or deformation.

This monograph is dedicated to the study of the change with time in the laws of strength and deformability of such elasto-visco-plastic materials as DSP-laminated plastics, plasticglasses and wood. Besides the fact, that all these materials are either artificial or natural organic compounds, they also have in common the fibrous structure and the presence of a well-defined skeleton and filler.

Wood has a fibrous structure with the fibers oriented along the axis of the trunk. An idealized scheme of its structure consists of a fibrous skeleton possessing elastic and plastic properties and an amorphous filler having viscous properties.

DSP-laminated plastic is a material obtained from thin sheets of husked birch plies treated with phenol- or cresol-formaldehyde synthetic resin. Constructional DSP-plastic belong to hard laminated plastics. The construction of the packets, and mutual arrangement and direction of wood fibers in the adjacent layers of the plies distinguish the several brands of DSP-plastics. DSP-plastic has a macroskeleton in the form of wood fibers of the plies and phenol-formaldehyde resins act as the filler of this skeleton. Hence, the complex composed of wood and resin, i.e., DSP-plastic, possesses well-defined properties of an elasto-visco-plastic body.

KAST-V glass textolite is a laminated sheet material obtained by hot pressing of properly laid layers of glass fibers of type ASTT(b), treated with phenol-formaldehyde resin, which is modified by polyvinylbutyral. Glass cloth is made of alumino-borosilicate staple glass fiber, from which alkali is removed by paraffin emulsion.

VFT-S glass textolite is also a laminated sheet material obtained by the method of hot pressing of properly laid layers of directly woven glass cloth of type ASTT(b), treated with modified VFT phenol-formaldehyde resin.

Anisotropic glass fiber material SVAM is a sheet material, obtained by the method of hot pressing of glass plies, treated with modified butvar, phenolformaldehyde, polyepoxide and other resins.

Plastic glasses have a macroskeleton in the form of fiber glasses of tissues or plies and a filler in the form of synthetic resins, which permit their classification in the same class of materials as DSP-plastics.

A number of investigations which were published earlier by the author along with F. P. Belyankin and G. I. Dybenko have been used in this monograph.

In the present work, the basic laws governing the change of the characteristics of resistibility, strength and deformability with time have been studied experimentally for plastic glasses, DSP-plastics and wood. Analytical relationships connecting the change of these characteristics with the conditions of the acting loads during compression, tension and bending have also been established. In contrast to previously published works, terminology has been changed here partially. Thus, the term "elastic limit" is used for "critical stresses"; the term "total limiting elastic deformation" for "critical elastic deformation"; the term "rate of development of limiting elastic deformations" for "critical rate of development of elastic deformations," and so on.

The author considers it his duty to express deep gratitude to F. P. Belyankin, member of the Academy of Sciences of the Ukrainian SSR for the systematic consultation and interest in this work, to Professor N. D. Zhudin for his valuable observations during the preparation of the monograph for publication, and also to the members of the Department of Strength of Materials of the Institute of Mechanics of the Academy of Sciences of the Ukrainian SSR, who helped considerably in carrying out the experimental part of the work and in systematizing the manuscript as a whole.

CHAPTER I

INVESTIGATION OF STRENGTH AND DEFORMABILITY OF LAMINATED PLASTICS AND WOOD

BASIC LAWS OF CREEP PROCESS

The influence of the time factor on the mechanical characteristics of laminated plastics and wood has been mentioned in a number of works published by the Soviet and foreign authors. However, the change in deformability, resistibility and strength were not considered together in these works. This led to the accumulation of cumbersome volume of experimental data, not fully reflecting the behavior of a material under load with the passage of time, and made it difficult to generalize the results.

Deformation with due consideration of the operating time of loading was experimentally studied first by Weber in 1835, by stretching a silken thread [112]. He established that, as a result of instantaneous application of a load, the silken thread continues to undergo deformation with time -- the deformation rate being high at the beginning and decreasing thereafter --, the deformation itself proceeding for a considerable time after the application of load. Analyzing the process of deformation with time, Weber explained it as being the consequence of rotation "of the smallest parts" in the elastically stressed body.

Subsequent investigations by Kohlrausch [113], Maxwell [114], Streĭnits, Butcher [110], Bach [108] and Kirpichev [50] established the significance of the influence of operating time of the load and the necessity of its consideration for determining the mechanical characteristics and for formulating the theory of strength and creep in real materials. It was established that the laws of formation and development of deformations with time under the action of a constant load for different materials have a common nature. If the load does not exceed a certain magnitude, the change in deformation with time follows an asymptotic law at a rate decreasing with time and the deformation itself,

without destroying the material, attains a constant value which depends upon the magnitude of the applied load. A large number of experimental works on different materials, for example, on concrete [21], plastic textolite [1], polymers [45, 46], wood [4, 38] etc., supporting this view, are known.

At loads higher than this magnitude, the process of development of deformation with time ends in the fracture of the material; the growth of deformation, starting from a certain value, follows a linear law, which is confirmed by the experimental data obtained for concrete [21], wood [4, 38] etc.

The process of recovery of strain after unloading (restoration of creep) has been investigated in a number of works [10, 20, 38, 56, 80]. It has been established that, if during loading the creep has a decaying nature, a complete restoration of the deformation takes place; on the other hand, if it does not have a decaying nature, then after unloading and subsequent sustaining, the plastic deformations are retained.

The processes of stress relaxation with time also have a common nature which has been confirmed by the experiments on wood [10], plastics [13] etc.

Along with the experimental studies of creep and stress relaxation, significant investigations have been done on the change in strength with time during prolonged action of a constant load.

The decrease in strength with time under the action of a constant load, observed by V.L. Kirpichev [50], was established by F.P. Belyankin [4, 5] for wood, which was afterwards confirmed by many others [20, 36, 38, 55, 59, 63]. The time dependence of the constantly acting stress till fracture was called the long-term resistance curve by F.P. Belyankin. The change in strength of the material with time follows an asymptotic law at a rate decreasing with time.

In recent years, investigations on prolonged strength of the polymers and other materials were conducted by S.N. Zhurkov [29-34], A.M. Borzdyka [15], G.M. Bartenev [2], N.A. Shkurko [91], Ya. B. Fridman [90], N.L. Leont'ev [60], B.N. Bykovskii [20], Yu. M. Ivanov [38] and B.A. Kiselev [51].

Thus, on the basis of wide experimental investigations, it may be considered that the elasto-visco-plastic materials under the action of a constant load have the following characteristics:

under a load not exceeding a certain value, the creep strain has an asymptotic nature and is reversible and in this case, the material does not fracture;

at loads exceeding the above value, the creep leads to fracture of the material; creep strains are not completely reversible and have a non-decaying nature;

strength of a material significantly decreases with time;

a decrease in stresses with time (relaxation) is observed under the action of a given constant deformation.

The experimental investigations on creep, stress relaxation and prolonged strength suggest a series of analytical relations representing these processes.

First attempts to express analytically the creep process in the elastic part of work of the material were made by Weber [112], who established a hyperbolic time dependence of the change in length of the specimen under the action of a constant load.

Somewhat later, Kohlrausch [113] confirmed the theoretical propositions of Weber, but he too only gave an empirical relation without establishing it with the help of mechanical or structural models of an elasto-viscous body.

The creep theory was significantly developed by Maxwell. In his paper [114], Maxwell developed the theory of deformability of real bodies considering them as a combination of two media: an ideally elastic medium obeying Hooke's law, and a viscous medium obeying Newton's law.

Maxwell suggested a relation between stresses and deformation in the form of a differential equation:

$$\frac{d\sigma}{dt} = E\frac{d\varepsilon}{dt} - \frac{\sigma}{T}, \qquad (1.1)$$

where σ denotes stresses; ε is relative deformation; E is modulus of elasticity and T is a constant having the dimension of time and termed as relaxation time by Maxwell.

Maxwell's idea of two media found wide application and a number of theories of deformation of a material with time have been proposed on its basis.

On the basis of Maxwell's idea of two media, Butcher [110] for the first time classified bodies into elasto-viscous, elasto-visco-plastic and visco-plastic. But this division of bodies is not quite correct, since external forces give rise to instantaneous elastic deformations, elasto-viscous deformations developing with time and plastic deformations also developing with time. Under different loads, these deformations may vary quantitatively for different materials. This observation had in fact led Butcher to the above-mentioned classification.

On the basis of the experiments of Weber, Kohlrausch and Streĭnits, Boltzman in 1874 proposed the theory of development of deformation with time, covering the entire process of deformation, beginning with the smallest and ending with the largest periods of time of deformation. He proposed a linear integral relation between stresses and deformation which is more general than the Maxwell's differential equation:

$$\sigma_{(t,\varepsilon)} = E\varepsilon_{(t)} - \int_{-\infty}^{t} f(t-\tau)\varepsilon_{(\tau)}\, d\tau. \qquad (1.2)$$

The function $f(t-\tau)$ characterizes the mechanical properties of the material and may be obtained experimentally.

Boltzman took the function $f(t-\tau)$ in the form of a hyperbola $\frac{A}{t-\tau}$ and obtained a relation, which does not sufficiently fit into the experimental data.

A more general theory of deformation with time, from which, in particular, the equation of Boltzman and the equation of Maxwell can be obtained, was proposed by Volterra [111].

Equations known as the Boltzman-Volterra relation have the form:

$$\sigma_{(t)} = E\varepsilon_{(t)} - \frac{\int_{0}^{t} K(t-\tau)\varepsilon_{(\tau)}\, d\tau}{T}, \qquad (1.3)$$

$$\varepsilon_{(t)} = \frac{\sigma_{(t)}}{E} - \frac{\int_0^t \Gamma(t-\tau)\,\sigma_{(\tau)}\,d\tau}{T}, \qquad (1.4)$$

where $K(t-\tau)$ is some function of inheritance, known as the coefficient of elastic after-effect, $\Gamma(t-\tau)$ is the coefficient of relaxation and T is the relaxation time.

Putting $\xi = \frac{t}{T}$ and $\zeta = \frac{\tau}{T}$ in the equations (1.3) and (1.4), we get

$$\sigma_{(\xi)} = E\varepsilon_{(\xi)} - \int_0^\xi K(\xi-\zeta)\,\varepsilon_{(\zeta)}\,d\zeta,$$
$$\varepsilon_{(\xi)} = \frac{\sigma_{(\xi)}}{E} + \int_0^\xi \Gamma(\xi-\zeta)\,\sigma_{(\zeta)}\,d\zeta. \qquad (1.5)$$

The functions $K_{(\xi)}$ and $\Gamma_{(\xi)}$ may be determined from the condition that the stresses $\sigma_0 = E\varepsilon_0$ applied to the specimen are constant or that the initial deformation $\varepsilon_0 = \frac{\sigma_0}{E}$ is constant.

The function $K_{(\xi)}$ is proportional to the rate of deformation at constant stresses and the function $\Gamma_{(\xi)}$ is proportional to the rate of change in stresses during constant deformation.

The functions $K_{(\xi)}$ and $\Gamma_{(\xi)}$ for any material may be obtained experimentally.

A. P. Bronskiĭ [16] obtained the function $\Gamma_{(\xi)}$ for several materials:

$$\Gamma(\xi-\zeta) = \bar{\sigma}\alpha e^{-\alpha(\xi-\zeta)^\alpha}(\xi-\zeta)^{\alpha-1}, \qquad (1.6)$$

where $\bar{\sigma}$ and α are some positive constants and $0 < \alpha < 1$.

A. R. Rzhanitsyn [79-84] considers the function $K_{(\xi)}$ in the form of a simple exponential function

$$K_{(\xi)} = Ae^{-\beta\theta}\theta^{-\alpha} \tag{1.7}$$

where $0 < \alpha < 1$ and $\beta > 0$.

Considering the laws of deformation in certain types of bodies, G. L. Slonimskiĭ [87, 88] arrived at the conclusion that for the required function the following equation can be used with high accuracy

$$f(t-\tau) = E_1 a K e^{-a(t-\tau)K}(t-\tau)^{K-1}, \tag{1.8}$$

where a, K and E are some positive constants.

Reviewing the mechanical modeling of the creep process in the elastic region of work of the material, A. R. Rzhanitsyn considers that with the increase in the number of elements in mechanical models as well as with the complication of their combination, it may be possible to arrive at complicated laws of deformation, which, however, do not lead to any qualitative changes in the nature of deformation.

A. R. Rzhanitsyn applied the following equation for expressing the law for time dependence of deformation:

$$H\eta\varepsilon + E\varepsilon = \sigma + \eta\sigma, \tag{1.9}$$

where E is the prolonged modulus of elasticity, H is the instantaneous modulus of elasticity and n is the relaxation time.

An analogous relation was proposed a little earlier by Bingam.

The law of the development of deformations with time under constant load is often taken in the form of a power function:

$$\varepsilon = a + bt^c. \tag{1.10}$$

This dependence is not satisfied by experimental data for long intervals of the time of action of the load [47, 89].

The expression for the process of the development of deformations with time in semilogarithmic coordinates [58], where the equation of the increase of deformation with time

$$\varepsilon = a + b \ln t \qquad (1.11)$$

gives a linear dependence of ε on $\ln t$, is extensively used. Semilogarithmic dependence is also satisfied by experimental data only in a limited interval of time. Many investigators [1, 10, 20, 26, 38, 73, 76, 79, 84] think that the laws governing the decrease in stresses at constant deformation as well as the increase in deformation at constant stresses are best expressed by exponential functions. The experimental results obtained during prolonged action of a constant load are most accurately satisfied by the exponential function:

$$\varepsilon = a + b e^{-\alpha \left(\frac{t}{T}\right)^c}. \qquad (1.12)$$

Considering a particular case when c = 1, we get a simpler expression for the deformation developing with time:

$$\varepsilon = a + b e^{-\alpha \frac{t}{T}} \qquad (1.13)$$

Interesting investigations on the behavior of the elasto-visco-plastic materials under the action of external load have been carried out by V.N. Bykovskiĭ [20]. Considering the material to be an elasto-viscous dispersed medium, V.N. Bykovskiĭ furnished theoretical assumptions and the basic dependence between stress and deformation with time. He obtained the differential equations of elasto-viscous textured and nontextured dispersed media in uniaxial stressed state, the integration of which gives the expression for total deformation as a function of time and stresses.

First of all, the names of F.P. Belyankin [5, 10], Yu. M. Ivanov [37, 43] and V.N. Bykovskiĭ must be mentioned amongst the investigators who studied the development of deformations and relaxation of stresses in the plastic region of work of the material.

It has been established during the investigations of the creep process that the rate of development of plastic deforma-

tions, depending on the magnitude of the stresses, is quite significant.

An analytical relation, mainly of nonlinear nature [62-65], connecting the rate of plastic deformation with the magnitude of stresses has been put forward in the literature:
in the form of a power function [65, 109]

$$V_s = k\sigma_s^n; \tag{1.14}$$

in the form of a hyperbolic cosine [65, 66]

$$V_s = C \sinh \frac{\sigma_s}{\alpha}; \tag{1.15}$$

in the form of several exponential functions [65]

$$V_s = ge^{\frac{\sigma_s}{n}} \tag{1.16}$$

and some other mixed functions.

Many investigators [5, 10, 20, 38, 78] consider that, in practice, it is possible to use the simple linear relation

$$\sigma_s = a + bV_s. \tag{1.17}$$

However, it may be pointed out that this relation satisfies the experimental data only within a comparatively narrow range of stresses and rates (mainly of smaller magnitudes).

INVESTIGATION OF PROLONGED STRENGTH

Prolonged strength of material is established on the basis of the relation between constant stresses and time interval till fracture, which is represented by a long-term resistance curve, generally obtained from experimental data.

Long-term resistance curve represents multiformity of the ultimate strength of a material as a function of the time of action of a constant load. According to experimental data, this curve has an asymptotic nature. Its asymptote determines the limiting value of long-term resistance.

The analytical relation between σ_s and t_{fr} is often expressed in the form of a linear logarithmic relation [29-34, 58, 110]:

$$\ln \sigma_s = a + b \ln t_{fr}. \qquad (1.18)$$

The equation (1.18) satisfies the experimental data in a limited interval of time [110].

Some investigators [5, 36] express the equation of the long-term resistance curve by a hyperbolic function:

$$\sigma_s = a + b t_{fr}^{-n}. \qquad (1.19)$$

The equation of the long-term resistance curve in the form of an exponential function satisfies most of the experimental results [10]:

$$\sigma_s = a + b e^{-\alpha t_{fr}}. \qquad (1.20)$$

Thus, it is most advisable to express the long-term resistance curve as well as the creep curve by exponential functions. It should be noted that a systematic study of prolonged strength has started comparatively recently and that the data for laminated plastics are still insufficient.

APPLICATION OF MECHANICAL AND STRUCTURAL MODELS IN THE STUDY OF RESISTIBILITY AND DEFORMABILITY OF ELASTO-VISCO-PLASTIC MATERIALS

In order to formulate a theory of deformability of elasto-visco-plastic materials, use is often made of mechanical models by regarding them as a combination of the elements imitating elastic, viscous and plastic properties.

Mechanical modeling is based on the similarity between the diagrams of deformation of a real body and a mechanical model [47, 80].

The simplest mechanical model of elasto-visco-plastic bodies (Bingam's model) consists of three elements connected in parallel or in series. Qualitatively, this model correctly represents most of the features of the deformation process, but

quantitatively, it almost always gives results differing from the experimental data. For greater conformity with experiment, complicated mechanical models are made by introducing more complex combinations of individual elements in the model of the body. Complication of the setups, as shown by A.R. Rzhanitsyn [80], does not introduce any qualitative differences, but analysis of the deformation process with the introduction of additional elements becomes considerably complicated.

An elasto-visco-plastic body is often represented as a combination of two media [20]. In this case, the process of redistribution of stresses between the two systems consisting of elastic and plastic elements is taken into consideration.

Structural models, based on the consideration of structural heterogeneity of material, are also used for establishing the creep process [35, 36]. In these models, the idea of structural heterogeneity is based on the hypothesis that the material is made of two phases. To the first phase are attributed particles whose prolonged strength is equal to zero, whereas particles whose prolonged strength has a different value from zero are attributed to the second phase. The creep process is regarded as a process, occurring during the redistribution of load between the weak particles of the first phase and the stronger particles of the second phase.

To explain the processes occurring in real bodies under load, idealized structural diagrams (one or more related to each other) are also used.

The idealized structural diagram of an elasto-visco-plastic material [10] contains two main components: the skeleton possessing elastic and plastic properties and the filler possessing viscous properties.

Taking the macrostructure of the material into consideration, it may be assumed that in plastic glass, for example, the glass fiber is the skeleton and the synthetic resin is the filler. Hence, the skeleton has fibrous structure, oriented in a definite manner depending on the type of the plastic glass.

Under the action of external forces, the elements of the skeleton get deformed in longitudinal and transverse directions and the filler offers a viscous resistance to the displacements

of the elements of the skeleton. The magnitude of the viscous resistance of the filler to the displacements of the elements of the skeleton depends on the coefficient of viscosity and the displacement rate of the elements of the skeleton in the filler.

Thus, the viscous filler obstructing the displacements of the elements of the skeleton offers support to the skeleton, i.e., increases its strength and hardness.

At the moment of its application, the external load is taken in by the skeleton and the filler, both of which deform together and give rise to an instantaneous elastic deformation. After the application of load and formation of an instantaneous elastic deformation, the process of unloading of the filler with time takes place at the cost of additional load to the skeleton, which leads to viscous deformation of the entire complex. This process, continuing with time, is mainly connected with redistribution of the internal elastic forces, arising in the skeleton and the filler. Thus, the viscous deformations of the entire complex depend upon the viscous resistance of the filler. If the applied load does not exceed a particular value, the displacements of the elements of the skeleton do not result in the failure of the latter and the viscous deformation of the entire complex, developing during an unlimitedly long period, is elastic. It is accepted that the maximum load, not causing the failure of the skeleton in course of an unlimitedly long time, corresponds to the limit of long-term resistance. Since in unlimitedly long time of action of the load, viscous resistance of the filler to the displacements of the elements of the skeleton comes to an end, the limit of long-term resistance is the maximum strength of the skeleton, i.e.

$$\sigma_{l.l.r.} = [\sigma_{str}]_{skeleton}$$

When the acting load gives rise to stresses greater than the limit of long-term resistance, local destruction of bonds, leading to the formation of irreversible plastic deformations, appears in the elements of the skeleton. The process of local failures of the elements of the skeleton continues with time and is completed after a limited interval of time, depending upon the magnitude of the load.

Let us consider the change in the carrying capacity of a material using an idealized diagram.

The applied external load is distributed between the skeleton and the filler according to their resistibilities.

Ultimate strength of the skeleton itself does not depend on the time of action of the load; the value of the viscous resistance of the filler, on the other hand, changes with time, exerting a supporting action of varying magnitude at different time intervals on the skeleton, thus changing the carrying capacity of the entire complex.

If the duration of the action of a constant load is small, the viscous resistance of the filler and hence its supporting action is high. Therefore, in order to break the entire complex, it is essential to apply a large load. With the increase of the time of action of a constant load, the supporting action of the filler is reduced and consequently the value of the breaking load decreases with time.

On the basis of the idealized diagram of an elasto-visco-plastic body, it is possible to establish the boundary conditions which must satisfy the equation of the long-term resistance curve representing the diversity of the limits of strength of a material as a function of the interval of time of action of the load till failure.

When the time $t \to \infty$, the viscous filler does not exert almost any supporting action on the skeleton and the carrying capacity of the material is determined only by the carrying capacity of the skeleton.

When $t \to 0$, the viscous filler exerts maximum supporting action on the skeleton and in this case, the carrying capacity of the material is determined by the carrying capacity of the entire complex, i.e., of the skeleton and the filler. The ultimate strength of the material attains a maximum, which is known as the limit of ultimate strength [10, 13, 102].

When $0 < t < \infty$, the strength of the material will be less than the limit of ultimate strength, but more than the limit of long-term resistance, because the supporting action of the filler on the elements of the skeleton is different for different time intervals.

According to experimental data, strength of a material decreases with the passage of time with a decreasing speed, the magnitude of which depends on the time of action of the breaking load.

When $t \to \infty$, the rate of change of strength with time tends to zero ($\frac{d\sigma}{dt} \to 0$).

When $t \to 0$, the rate of change of strength with time tends to infinity ($\frac{d\sigma}{dt} \to -\infty$).

Thus, the boundary conditions for the curve of prolonged resistance may be written in the following form:

$$\left.\begin{array}{lll} \text{when} & t \to 0 \quad \sigma \to \sigma_{str}^{lim}, & \frac{d\sigma}{dt} \to -\infty, \\ \text{when} & t \to \infty \quad \sigma \to \sigma_{l.l.n.}, & \frac{d\sigma}{dt} \to 0, \end{array}\right\} \qquad (1.21)$$

where σ_{str}^{lim} is the limit of ultimate strength of the material; $\sigma_{l.l.n.}$ is the limit of long-term resistance and t is the time till failure under the action of a constant load.

The curve of prolonged resistance is represented by the coordinates "ultimate strength -- time till fracture" in the form of a monotonic concave line of asymptotic nature (Fig. 1). The asymptote to this curve cuts, on the stress axis, an intercept which corresponds to the limit of long-term resistance.

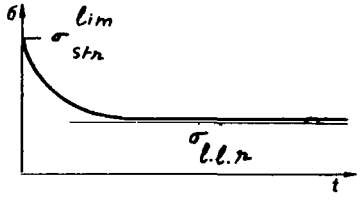

Fig. 1. Long-term resistance curve.

According to the abovementioned boundary conditions (1.21), the equation of the long-term curve may be represented as

$$\sigma = a + b\, e^{-\alpha_1 \left(\frac{t}{t_2}\right)^{C_1}}, \qquad (1.22)$$

where a and b are coefficients depending upon the strength of the material; c_1 is a constant depending upon the rate of change in the breaking stress with time; t is the time till failure; t_2 is the time base for the prolonged resistance curve at which the ultimate strength, changing with time, attains a value higher than the limit of long-term resistance by L % (allowance for determining the strength); α_1 is a constant depending upon the allowance in stress for determining the time base of the tests (t_2).

The coefficients a and b are determined from the boundary conditions (1.21):

$$a = \sigma_{l.l.n.}, \quad b = \sigma_{str}^{lim} - \sigma_{l.l.n.}.$$

Then the equation of the prolonged resistance curve (1.22) becomes:

$$\sigma = \sigma_{l.l.n.}\left[1 + \left(\frac{\sigma_{str}^{lim}}{\sigma_{l.l.n.}} - 1\right) e^{-\alpha_1 \left(\frac{t}{t_2}\right)^{c_1}}\right]. \qquad (1.23)$$

All the abovementioned coefficients except α_1 may be obtained by processing the experimental data statistically. The functions, differing from the exponential (1.23), used by various investigators for representing the prolonged resistance curve, do not satisfy the boundary conditions (1.21) either completely or partially.

Let us consider the creep process under the prolonged action of a constant load, using an idealized structural diagram of elasto-visco-plastic material.

The external load is taken in by the skeleton as well as by the filler at the moment of its application.

Irrespective of the fact whether the load is applied instantaneously or gradually from zero to its final value, the joint deformation of the skeleton and the filler leads to the formation of an instantaneous elastic deformation, which increases with a rate, equal to the rate of application of the load during the process of loading.

In the case of a constant load, acting for a long period, the instantaneous elastic deformation arises only in the process of applying or changing the load. After the application of the load

and the formation of the instantaneous elastic deformation, the viscous deformation of the filler develops under the action of the part of the external load which is taken in by the filler. In a statically indeterminate system, consisting of the skeleton and its filler, the viscous deformation of the filler is followed by the process of unloading of the filler with time, arising as a result of the process of additional loading of the skeleton with time.

Along with the process of redistribution of the internal elastic forces under the action of the load applied to the skeleton, its elements get deformed along and transverse to their axis. The transverse deformation of the elements of the skeleton is related to the change in their configuration and the displacement in the viscous medium of the filler gradually decaying with time. Deformation of the skeleton in the direction of the action of the external forces, which is related to the change in configuration of its elements and transverse displacements in the viscous medium of the filler, is the source of the formation of elastic deformations of the skeleton, developing with time under the action of a constant prolonged load.

Thus, the formation and growth of elastic deformations with time under the action of a constant load is determined by the viscous resistance of the filler.

The load, falling on the skeleton at the time of loading, mainly causes deformation of the elements of the skeleton along the axis, i.e., when $t \to 0$, viscous deformation is absent and the total deformation consists only of the instantaneous elastic deformation

$$\varepsilon_{\text{total}} = \varepsilon_{\text{inst.}}$$

The elasto-viscous deformation arises after the formation of the instantaneous elastic deformation on loading. When there is no failure of the elements of the skeleton, this deformation grows with time at a rate decreasing to zero, i.e., when

$$t \to \infty, \frac{d\varepsilon_{e.v.}}{dt} \to 0.$$

According to experimental data, the instantaneous deformation arising at the time of loading is completely restored on unloading, and hence is elastic. Elasto-viscous deformation, developing with time under the action of a constant load, is not

fully restored immediately after unloading, i.e., it is residual; but after a time interval, corresponding to the time interval of the action of load, this deformation is completely restored, i.e., it is factually elastic.

If the failure of the elements of the skeleton does not take place during loading, the configuration of the elements of the skeleton on complete unloading is restored to its original form under the action of the internal elastic forces. The displacements of the elements of the skeleton, caused by the change in the configuration of the skeleton during its unloading, are connected with the appearance of the viscous resistance of the filler impeding such displacements. As the experiments show, the recovery after unloading follows the same law as on loading, the recovery being equal in magnitude to the corresponding deformation on loading.

Let us consider the boundary conditions which must satisfy the equation of the creep curve in the elastic region of work of the material at prolonged action of a constant load giving rise to stresses less than the limit of the long-term resistance.

When $t \to \infty$, the viscous deformation is fully developed and the total deformation is equal to the sum of the instantaneous elastic and maximum elasto-viscous deformations. In this case, the total deformability of the entire complex is characterized by the prolonged modulus of elasticity E_{prol}. Now the rate of change in total deformation with time tends to zero, since the viscous resistance offered by the filler to the displacements of the elements of the skeleton disappears completely and the skeleton gets maximum deformation for the given stress. Hence the prolonged modulus of elasticity characterizes the elastic deformability of the skeleton.

When $t \to 0$, the viscous properties of the filler do not appear and the total deformation is equal to the instantaneous elastic deformation of the skeleton and the filler, which is characterized by the instantaneous modulus of elasticity E_{inst}. In this case, the filler offers maximum resistance to the displacements of the elements of the skeleton which diminishes rapidly with time. Elasto-viscous deformation at $t \to 0$ changes with time at a rate $\frac{d\varepsilon_{e.v.}}{dt} \to \infty$, and its value $\varepsilon_{e.v.} \to 0$, because at the time $t \to 0$, it fails to develop in the bulk of the body under deformation.

Thus the boundary conditions for the creep curve at a stress less than the limit of long-term resistance are as follows:

when

$$t \to 0 \qquad \varepsilon_{total} \to \varepsilon_{inst} = \frac{\sigma}{E_{inst}}, \quad \frac{d\varepsilon_{total}}{dt} \to \infty,$$

when

$$t \to \infty \qquad \varepsilon_{total} \to \varepsilon_{prol} = \frac{\sigma}{E_{prol}}, \quad \frac{d\varepsilon_{total}}{dt} \to 0. \qquad (1.24)$$

Equation of the creep may be represented in the form:

$$\varepsilon_{total} = \varepsilon_{inst} + \int_0^t f(v)\, dt, \qquad (1.25)$$

where under the integral sign, we have a function of the rate of growth of elasto-viscous deformation with time.

Creep curves under different loads are shown schematically in Fig. 2. The lower curves (1, 2) depict the creep process in the elastic region of work of the material, whereas the upper ones (3-5) are for the elasto-plastic region. Diagrams corresponding to these curves, showing the change in the rate of development of deformation with time are shown in Fig. 3, where the monotonically decreasing curves correspond to creep diagrams in the elastic region of work of the material and the straight horizontal lines are for the plastic region of work.

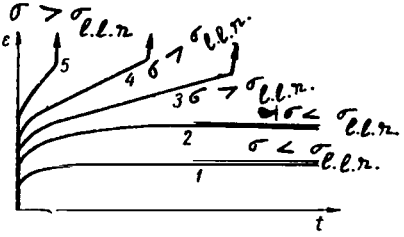

Fig. 2. Curves of creep under prolonged loading.

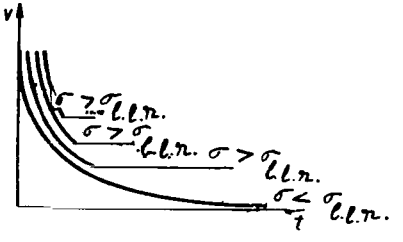

Fig. 3. Curves of change in creep rate under prolonged loading.

Equation of creep curve in the elastic region of work of the material, according to the abovementioned boundary conditions (1.24), may be represented in the form

$$\varepsilon_{total} = a + be^{-\alpha\left(\frac{t}{t_1}\right)^c} \qquad (1.26)$$

where a and b are coefficients depending on the nature of deformability of the material; c is a constant depending on the change in the creep rate with time; t is the time of action of the constant load; t_1 is the time base for creep curve, during which the total elastic deformation, changing with time, attains a value which is less by L % (allowance for determining the deformations) than the value of the total deformation when the time taken for its development is unlimited and α is a constant depending on the allowance in deformation for determining the time base of test. Coefficients a and b are determined by boundary conditions (1.24)

$$a = \frac{\sigma}{E_{prol}}; \quad b = \frac{\sigma}{E_{inst}} - \frac{\sigma}{E_{prol}}.$$

Then the equation of creep curve becomes

$$\varepsilon_{total} = \frac{\sigma}{E_{prol}}\left[1 - \left(1 - \frac{E_{prol}}{E_{inst}}\right)e^{-\alpha\left(\frac{t}{t_1}\right)^c}\right]. \qquad (1.27)$$

All the abovementioned coefficients with the exception of α may be obtained by statistical processing of the experimental data.

It should be pointed out that functions analogous to (1.27) have also been used by other investigators [16, 38] for processing the experimental data.

In the plastic region of work of the material, the rate of development of deformation with time is constant for given stresses, but changes with variation in the magnitude of the stresses (see Fig. 3, horizontal lines). Therefore it is necessary to know the dependence between the values of constant stresses and the rate of plastic deformations corresponding to them.

In the case of a constant load (see Fig. 2, curves 3-5) corresponding to the stresses above the limit of long-term resistance, instantaneous elastic, elasto-viscous and plastic deformations are observed.

Formation of plastic deformations may be interpreted as the development of local failures of the elements of the skeleton of the idealized structural diagram of the material. Local failures appear not only as a result of local failure of the connection between the elements of the skeleton under shear but also as a result of local loss of stability of its elements and also as a result of local failure of connections under large longitudinal and transverse local deformations of the elements of the skeleton. In the course of time, further growth of plastic deformations under constant stress takes place as a result of new local failures of the elements of the skeleton and the development of the old ones.

The growth and unification of local failures of the elements of the skeleton lead to the separation of its parts, which is the actual cause of failure. This process does not occur instantaneously but proceeds with the passage of time at an increasing rate, in most of the cases rapidly, which actually permits the failure to be considered as practically instantaneous.

The value of the constantly acting stresses which cause the development of plastic deformations, and consequently, the failure of the material, may vary in the interval from $\sigma \to \sigma_{l.l.r.}$ at the time $t \to \infty$ to σ_{str}^{lim} at $t \to 0$. If the stresses $\sigma \to \sigma_{str}^{lim}$, the failure of the material takes place during time $t \to 0$ and the plastic deformation cannot develop ($\varepsilon_{plast} \to 0$) despite the fact that its rate $v_{plast} \to \infty$, i.e., there takes place a brittle fracture with the formation of only instantaneous elastic deformations.

The experimental data indicate that with a decrease in the stresses, the plastic deformation develops with time at a decreasing rate. Moreover, since the failure of the material takes place after a finite time interval, the plastic deformation has a finite value.

With the decrease in the stresses to a value equal to the limit of the long-term resistance, the rate of development of plastic deformations and their magnitude decrease to zero [5, 10, 20, 38, 97].

Thus, the boundary conditions of the relation between the value of constant stresses and the rate of development of plastic deformations are as follows:

when $\sigma \to \sigma_{str}^{lim}$ $v_{plast} \to \infty$

when $\sigma \to \sigma_{l.l.n.}$ $v_{plast} \to 0$. (1.28)

The relation between the magnitude of stresses and plastic deformation must satisfy the condition:

when $\sigma \to \sigma_{str}^{lim}$ $\varepsilon_{plast} \to 0$,

when $\sigma \to \sigma_{l.l.n.}$ $\varepsilon_{plast} \to 0$. (1.29)

Since at time intervals $t \to 0$ and $t \to \infty$ of the action of constant load causing failure of the material, plastic deformations do not take place and the brittle fracture occurs only because of elastic deformations, the dependence between stresses and deformations during failure must satisfy the condition:

when $t \to 0$ and $\sigma \to \sigma_{str}^{lim}$ $\varepsilon_{fr} \to \frac{\sigma_{str}^{lim}}{E_{inst}}$,

when $t \to \infty$ and $\sigma \to \sigma_{l.l.n}$ $\varepsilon_{fr} \to \frac{\sigma_{l.l.n.}}{E_{prol}}$. (1.30)

The abovementioned boundary conditions for the relation between the change in strength and deformability with time at prolonged action of a constant load make it possible to select appropriate forms of the functions representing these relations. These conditions are obtained on the basis of an idealized struc-

tural diagram of the elasto-visco-plastic material and analysis of the experimental data of materials with clearly expressed elastic, viscous and plastic properties.

Besides this, the idealized structural diagram permits an explanation of the change in strength and deformability depending on the change in hydrothermic state of the material [10, 13]. With decrease in humidity and temperature of the filler as a consequence of the change in the hydrothermic state of the material, skeleton-like elements separate out in the filler, which help in strengthening the main skeleton of the idealized structural diagram. These elements are additional connections superimposed on the main skeleton. With increase in humidity and temperature, the additional connections may be eliminated to a greater or lesser extent.

CHAPTER II

EXPERIMENTAL INVESTIGATION OF STRENGTH AND DEFORMABILITY OF LAMINATED PLASTICS AND WOOD UNDER PROLONGED ACTION OF CONSTANT LOAD

AIMS OF INVESTIGATION

In order to explain the laws of change in strength and deformability of laminated plastics and wood taking the time factor into consideration, experiments were conducted mainly in two directions:

1. Investigation of the change in strength under the action of constant static load of different magnitudes.

2. Study of creep at the same load.

The first series of experiments were conducted to obtain the curves of prolonged resistance and to determine the main parameters characterizing the change in strength with time(limit of long-term resistance, limit of ultimate strength, base of the tests, etc.).

The second series of investigations were carried out in order to experimentally verify the creep relationships at different loads and to determine the main parameters characterizing creep (instantaneous and prolonged modulus of elasticity, base of the tests, etc.). Besides this, the relation between the rate of plastic deformation and stresses, proportionality laws for instantaneous elastic and elasto-viscous deformation, etc. were also investigated.

Experiments were carried out in tension and compression state on plastic glasses, laminated plastics and wood, taking into view the anisotropy of the materials at normal temperature and humidity.

Since the determination of some characteristics of materials directly from prolonged tests was difficult (e.g., range of ultimate strength, instantaneous modulus of elasticity), these were determined from transitory loading at a constant rate or from stepwise tests.

TESTING MACHINES AND APPARATUS

Transitory and prolonged tests on laminated plastics and wood were conducted on two types of machines:

1. Universal hydraulic machines of the type "Emery Baldwin", and

2. Lever mechanisms for testing specimens at prolonged action of the load [14].

The universal hydraulic testing machines of the type "Emery Baldwin" permit one to conduct tests at a given rate of loading in a wide range of rates and also at a given rate of deformation, if necessary.

The lever mechanisms permit testing of laminated plastics and wood at prolonged action of a static constant load and at a given rate of loading. The force is produced by the mechanism of single lever transmission of load with arm ratio 1:2 and 1:20, which makes it possible to test laminated plastics and wood at maximum loads of 400 and 4000 dkn. The abovementioned stages of loading are arranged by transposing the bearing prisms of the lever. The force is measured directly by the weight of the applied load, transferred to the specimen through the lever [9, 11, 13]. The machine produces a tensile force and is provided with suitable clamps and also with a system for converting it into compression and bending [14].

Absolute deformations were measured by indicators and Gugenberger tensometers and relative deformations were measured with the help of wire type ohmic pickups by SD-1 static deformometers [64].

Absolute bending deformations were measured by an indicator fitted on a /7 -shaped clamp pressed to the surface of

the sample by two rubber rings. One end of the clamp rests on the cylindrical surface of the specimen whereas the other end rests on the prismatic surface. The clamp is placed within the limits of the working part and have a 75 mm base [14]. With the help of this device, the deflection at the mentioned length of the base is measured by the indicator. The adopted device eliminates the influence of crumpling of the material and of displacements of the specimen on the magnitude of deflection during the deflection measurement.

SET-UP FOR CUTTING THE SPECIMENS, SELECTION OF THEIR SHAPES AND DIMENSIONS

Specimens for the tests were prepared from blanks, cut from sheets of laminated plastics. The blanks were cut and specimens were prepared on metal cutting machine tools (mainly milling machines).

In order to reduce the influence of the surface treatment on the value of mechanical characteristics of laminated plastics for all the specimens under investigation, the right-angled shape of the transverse section with the orientation of the faces of the specimen along the principal planes of anisotropy was accepted.

In order to reduce the influence of face friction and loss of stability, specimens for compression (mainly of size 12 x 12 x 18 mm) had the shape of a parallelopiped with the ratio of the cross section to height 1:1:1.5. These specimens were pre-prepared with an allowance of not more than ± 0.1 mm along the height and along the sides of the cross section. While preparing the specimen, deviation from parallelism of the loaded faces was eliminated and the mutual perpendicularity of all its faces was maintained. While selecting the specimens for test, specimens having deviations in dimensions beyond the required allowances as well as those with insufficient surface finish were rejected. Since laminated plastics and wood possess triaxial or biaxial anisotropy, the specimens for compression test were cut out of a slab along all principal axes of anisotropy.

Specimens for tensile tests had wedge-shaped heads, made according to the shape of the clamps [14]. Such fixing of the specimens into the clamps eliminates the eventual eccentric application of load along the ends of the specimens

due to the displacement of the wedges relative to each other and also fracture of the specimen beyond the limits of the working part as is often observed in tests on plane specimens in a two-wedged clamp. The working part of the specimen during tensile tests is connected to the wedge-shaped head by means of recesses having a smooth transition near the head.

The allowance on the working part of the specimen for tensile tests was taken as ± 0.1 mm for the cross section and ± 0.5 mm for the length.

Tests on bending were conducted on specimens with supported loading heads. Such specimens eliminate the influence of crumpling of the working part at the points of application of load as well as at the supporting points of the specimen. The specimens were cut from slabs in such a way that in one case the plane of the bending was parallel to the plane of the ply or the tissue and, in the other case, perpendicular to it [14]. The allowances in the dimensions of the cross section of the working part of the specimens were ± 0.1 mm.

While testing wood for tension along the fibers, specimens with wedge-shaped heads were used, whereas for bending, specimens with supported loading heads, and for compression, standard specimens were used.

DETERMINATION OF VOLUMETRIC WEIGHT AND MOISTURE

Volumetric weight (quotient of division of weight of the specimen by its volume) of laminated plastics and wood (in the condition in which they were supplied) was determined on specimens (300 specimens of each type), selected for compression test. The volume of the specimen was determined by linear dimensions, measured with an accuracy of ± 0.1 mm; the weight was determined with an accuracy of ± 0.1 g.

The absolute moisture in laminated plastics and wood was also determined for specimens for compression test. The specimens, in the initial condition, were weighed with an accuracy up to ± 0.001 g, after which they were placed in a thermostat having a regulator capable of controlling the temperature with an accuracy of ± 2°C. The samples were kept in the thermostat for two

days at a temperature of 100-105°C and then cooled down in a dessicator to a temperature of 20°C and weighed with the same accuracy as in the initial condition. After weighing, the specimens were again placed in the thermostat and kept for the same time and at the same temperature. This process was continued till the weight of the specimens was constant, when they were considered to be absolutely dry. Absolute moisture, i.e., the ratio of the difference in weights of the specimen in the initial state and in the absolute dry state to its weight in the absolutely dry state, was determined with an accuracy of 0.01%. The results of the determination of the volumetric weight and the humidity of laminated plastics and wood are presented in Table 1.

Table 1

Material	Volumetric weight, g/cm^3			Absolute humidity, %		
	Average	Maximum	Minimum	Average	Maximum	Minimum
KAST-V	1.75	1.78	1.67	1.5	1.6	1.3
VFT-S	1.80	1.85	1.78	1.7	3.1	0.7
SVAM	1.77	1.85	1.69	0.6	0.8	0.4
DSP-B	1.29	1.37	1.14	2.0	2.9	1.5
DSP-V	1.29	1.32	1.27	4.0	4.5	3.7
DSP-G	1.31	1.33	1.28	3.3	3.5	2.9
Pine	0.54	0.55	0.53	10.0	12.0	8.0
Oak	0.75	0.77	0.72	11.0	12.0	9.0
Beech	0.62	0.65	0.59	10.0	12.0	9.0

DETERMINATION OF LIMIT OF ULTIMATE STRENGTH

The ultimate strength, i.e., the maximum value of strength of a material when the time of action of a constant breaking load $t \rightarrow 0$, is difficult to determine directly from prolonged tests, since it is practically impossible to predict the magnitude of the load at which the failure time $t \rightarrow 0$. Hence, it is convenient to determine this very important characteristic of strength of material from tests with high rate of loading or deformation, which gives practically the value of the strength of material when $t \rightarrow 0$ if the rate of test $v_{load} \rightarrow \infty$.

Tests with different rates of loading or deformation mainly give an idea about the strength of material during short intervals of time, characterize maximum strength and permit to obtain the value of the limit of ultimate strength.

Investigation of the influence of the rate of loading on the ultimate strength in the case of compression and tension was conducted on specimens by means of a 30-ton hydraulic machine of the type "Emery Baldwin" in the range of loading rates of 150-30000 dkn/min.

For obtaining one curve showing the variation of ultimate strength with the rate of loading, 8-10 specimens of laminated plastics and wood were tested at each given rate.

While processing the experimental data, in order to eliminate the effect of dimensions of the cross sections of the specimens, the rate of loading (dkn/min on a specimen) was converted into the rate of increase in stresses (dkn/cm^2, min), which was obtained by dividing the given rate of loading by the area of the cross section of the specimen.

The diagrams of the change in ultimate strength with rate were drawn on the basis of the average value of the ultimate strength at each given rate of increase in stresses. For example, Fig. 4 shows the curves of the change in ultimate strength with the rate of increase in stresses at t =18 to 20°C for plastic glass SVAM, KAST-V and laminated plastic DSP-B. It is seen that the variation of the ultimate strength with the rate of increase of stresses has an asymptotic nature. At small rates of increase in stresses, their change significantly affects the magnitude of ultimate strength and at higher rates, only slightly.

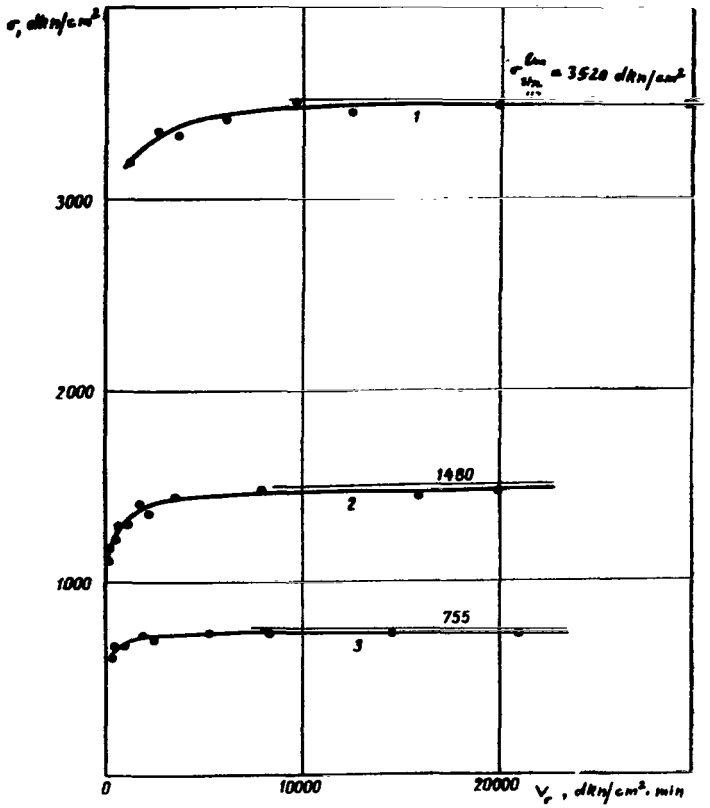

Fig. 4. The curve of variation of ultimate strength with the rate of increase in compression stresses.

1 -- SVAM along the layers of the ply; 2 -- DSP-B perpendicular to the layers of the ply; 3 -- KAST-V along the warp.

The asymptote to the curve of variation in σ_{str} with v_σ determines the limit of ultimate strength. These curvilinear relationships also permit us to establish the test rate at which the value of the ultimate strength depends mainly on the straggling of the experimental data and does not depend on the time factor during test.

On the asymptotic branch of the curve $\sigma_{str} - v_\sigma$, it is possible to find out the point, that determines the rate of tests above which the change in the rate of increase in stresses insignificantly affects the magnitude of ultimate strength. This rate of increase in stresses is called the optimum rate.

The value of the limit of ultimate strength and that of the optimum rate of increase in stresses are determined by statistical analysis of the experimental data.

While determining the limit of ultimate strength and the optimum rate of increase in stresses on the asymptotic part of the curve, the variation in σ_{str} with v_σ may be taken in the form of a hyperbola

$$\sigma = a - \frac{b}{v_\sigma},$$

where a and b are coefficients obtained by statistical analysis of the experimental data.

The value of the optimum rate of increase in stresses may be determined from the above equation with a 5% allowance in stresses from the value of the limit of ultimate strength:

$$v_{opt} = \frac{b}{0.05a}.$$

The result of determination of the limits of ultimate strength and optimum rates of increase in stresses for plastic glasses, DSP-laminated plastics and wood subjected to compression and tension at t = 18 to 20°C are accordingly given in Tables 2 and 3 respectively.

Table 2

Material	Direction of action of force	Limit of ultimate strength, dkn/cm^2	Optimum rate of increase in stresses, dkn/cm^2. min
VFT-S	Along the warp	1830	1700
	Along the weft	1200	580
	Perpendicular to the layers	3850	1245
KAST-V	Along the warp	755	850
	Along the weft	720	1750
	Perpendicular to the layers	3610	1900

Contd.

1	2	3	4
CVAM	Along the layers of ply	3520	2930
	Perpendicular to the layers of ply	2545	1240
DSP-B	Along the fibers of lining along the ply layers	1940	340
	Transverse to the fibers of the lining along the ply layers	1240	500
	Perpendicular to the layers of ply	1480	2200
DSP-V	Along the ply layers along the fibers of the lining	1480	400
	Perpendicular to the ply layers	2860	1100
DSP-G	Along the ply layers along the fibers of the lining	1560	1000
	Perpendicular to the ply layers	2565	1900
Pine	Along the fibers	760	-
Oak	"	728	-
Beech	"	650	-

Table 3

Material	Direction of action of force	Limit of ultimate strength dkm/cm^2	Optimum rate of increase in stresses, dkn/cm^2 min
KAST-V	Along the warp	1490	2030
	Along the weft	1320	1410
BFT-S	Along the warp	2990	1880
	Along the weft	1630	780
DSP-B	Along the ply layers along the lining	2860	380
DSP-V	-do-	1480	1820
DSP-G	-do-	1330	600
Pine	Along the fibers	1115	-
Oak	-do-	1210	-
Beech	-do-	1770	-

As is seen, in some cases the optimum rates of test significantly differ from the GOST. It should also be mentioned that in an overall complex investigation of new materials, mainly on specimens differing from the standard ones, it is essential to observe strictly the condition of similarity in selecting the test rates for obtaining comparable results for the purpose of generalization.

DETERMINATION OF LIMIT OF LONG-TERM RESISTANCE

The limit of the long-term resistance -- the minimum value of strength of a material at indefinitely long period of

action of a constant breaking load -- is an important mechanical characteristic of a material. The most reliable method of determining the limit of long-term resistance is its determination from the prolonged resistance curve, obtained directly from the tests with a sufficiently high time base.

For obtaining the appropriate time interval of observations as the test base, the result of prolonged tests of material with sufficient time base (up to one year or more) are utilized. Using the results of experiment with the bases 1, 2, 3 months, we obtain the equation of the prolonged resistance curve in the accepted form of the function and the magnitude of the limit of long-term resistance by analytical means by the method of least squares for each complex of experimental points corresponding to each of the abovementioned time bases. The results of calculations show that the magnitude of the limit of long-term resistance and its average quadratic deviation decreases sharply in the beginning with an increase in the time base and then asymptotically approaches its limiting value. If it is considered that the experimental value of the mechanical characteristics generally deviates by 5% from the average value, then calculated values of average quadratic deviates by 5% for the time base of 1.5-2 months. The same will be the case with limited time interval of observations, which was accepted in the experiments for determining limits of long-term resistance [10, 13].

The determination of the limit of long-term resistance from the curve of prolonged resistance requires the extrapolation of the curvilinear relation to an infinitely long time. In reality, the determination of the limit of long-term resistance is carried out on the basis of the experimental data obtained within the limits of a limited time interval.

Let us call the limit of long-term resistance at restricted time base of test as the restricted limit of long-term resistance. Thus, the determination of this limit is connected with the extrapolation of the prolonged resistance curve to infinitely long term with least deviation from such experimental data which have been obtained during the time interval of tests. The restricted limit of long-term resistance obtained in such a way will be called in future the limit of long-term resistance.

The prolonged resistance curve is obtained as a result of the test of a series of specimens loaded by different values of

a constantly acting prolonged load. The value of the load is in the range of 90-30% of the load, corresponding to the limit of ultimate strength.

For each specimen, broken under the action of a constant load, stresses and time till failure are determined.

Experimental prolonged resistance curves were obtained on the basis of the experimental data in the coordinates "stresses -- time till failure".

Tests on prolonged strength were generally conducted for a period of 1-2 months. It should be noted, however, that the sharpest fall of strength with time was observed in the course of 3-5 days and later the decrease in strength with time was insignificant.

The specimens, which did not undergo fracture in course of the adopted time base of experiments with stresses less than the minimum failure stresses, were not taken into consideration while plotting the prolonged resistance curves. The specimens, which did not undergo fracture in the course of this time interval with stresses higher than the minimum failure stresses, were taken into consideration for plotting the prolonged resistance curves provided the time till breaking was infinitely long.

To simplify the subsequent analysis of the experimental data for determining the equation of the prolonged resistance curve and for finding the limit of long-term resistance, the value of the limit of ultimate strength was included in the experimental data.

Limit of long-term resistance $\sigma_{l.l.r.}$, time base t_2 and the coefficient c_1 in the equation of the prolonged resistance curve (1.23) can be found from statistical analysis of the experimental data.

In agreement with the boundary conditions (1.21), coefficient c_1 lies within the limits $0 < c_1 < 1$.

For simplifying the determination of the value of the limit of long-term resistance by the method of least squares, c_1 may be assumed to be equal to 1. This assumption will distort the

prolonged strength curve in the region of small time intervals of action of a constant load to some extent, but will considerably simplify the determination of the value of limit of long-term resistance. In this case, the equation of prolonged resistance curve will have the form

$$\sigma = a + be^{-\alpha \frac{t}{t_2}},$$

or, since $b = \sigma_{str}^{lim} - a$,

$$\sigma = a + (\sigma_{str}^{lim} - a)\, e^{-\alpha \frac{t}{t_2}} \qquad (2.1)$$

Equation (2.1) has only two constants: a and α/t_2. The constant α/t_2 is determined, at the time of selecting the equation of prolonged resistance curve, from the minimum of the mean quadratic deviation.

In the case of any particular value of α/t_2, the constant a is obtained from the fact that

$$\sum_1^n \left[\sigma - \sigma_{str}^{lim} e^{-\alpha \frac{t}{t_2}} - a\left(1 - e^{-\alpha \frac{t}{t_2}}\right)\right]^2$$

has the minimum value, i.e.

$$\frac{d \sum_1^n \left[\sigma - \sigma_{str}^{lim} e^{-\alpha \frac{t}{t_2}} - a\left(1 - e^{-\alpha \frac{t}{t_2}}\right)\right]^2}{da} = 0,$$

which gives $\sigma_{l.l.n.} = a$, where

$$a = \frac{\sum_1^n \left(1 - e^{-\alpha \frac{t}{t_2}}\right)\sigma - \sigma_{str}^{lim} \sum_1^n e^{-\alpha \frac{t}{t_2}}\left(1 - e^{-\alpha \frac{t}{t_2}}\right)}{\sum_1^n \left(1 - e^{-\alpha \frac{t}{t_2}}\right)^2} \qquad (2.2)$$

Putting the value of α/t_2 from 0.1 to 10, we obtain the coefficient a for each value of α/t_2.

From values of σ obtained experimentally and selected on the basis of the equation of prolonged resistance curve, we obtain the value of the deviation δ and the mean quadratic deviation:

$$\Delta = \sqrt{\frac{\sum \delta^2}{n-1}},$$

where n is the number of experimental values for each of the adopted values of α / t_2.

We take those values of α / t_2 and a as the most probable ones which give the minimum value of Δ.

In order to find the coefficient c_1 characterizing the rate of change in ultimate strength with time from the prolonged resistance curve, statistical method has also been used, taking into consideration that in the equation of prolonged resistance curve, all parameters except c_1, are determined by the method stated above. In this case, the problem leads to the determination of one constant value by the statistical method. For this, we write the equation (1.22) in the form

$$\ln \frac{\sigma - a}{b} = -\alpha \left(\frac{t}{t_2} \right)^{c_1} \tag{2.3}$$

According to the method of least squares, we determine the value of c_1 from the condition

$$\frac{d}{dc_1} \sum_{1}^{n} \left[\ln \frac{\sigma - a}{b} + \alpha \left(\frac{t}{t_2} \right)^{c_1} \right]^2 = 0 \tag{2.4}$$

In Figs. 5 - 7, prolonged resistance curves at normal temperature (t = 18 to 20°C) are presented as examples of compression and tension of plastic glasses, DSP-plastic and wood. Experimental points, limits of ultimate strength and limit of long-term resistance, and also the selected prolonged resistance curve, which show minimum deviation from the experimental data, are shown here. As seen, the sharpest fall in strength is observed in the course of the first 2-3 days.

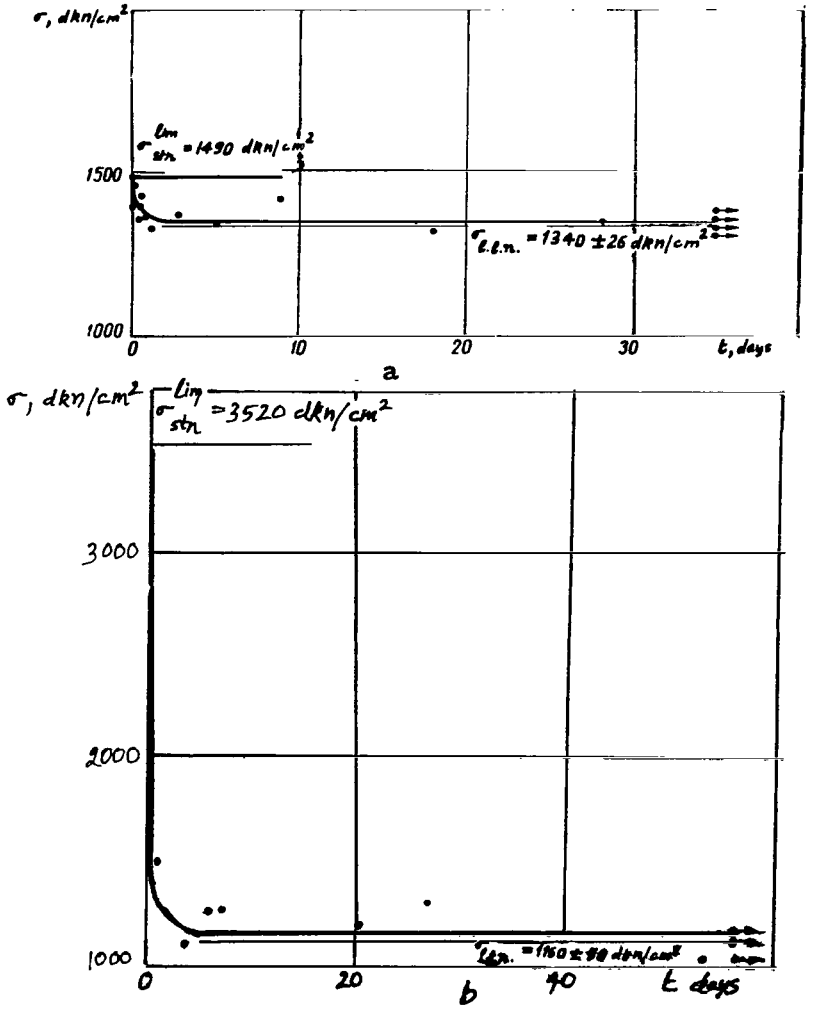

Fig. 5. Prolonged resistance curves.
a - tension of KAST-V along warp; b - compression of SVAM along the ply.

The results of prolonged tests of laminated plastics and wood are given in Table 4, where limit of long-term resistance $\sigma_{l.l.n.}$, time base t_2, coefficient c_1 and the ratio of the limit of long-term resistance to the limit of ultimate strength presented in Tables 2 and 3, are also included.

As is obvious from Table 4, the decrease in strength with time under prolonged action of a constant load, in most of the cases, is within 45-55%, which indicates the significant influence of time factor on the strength of a material.

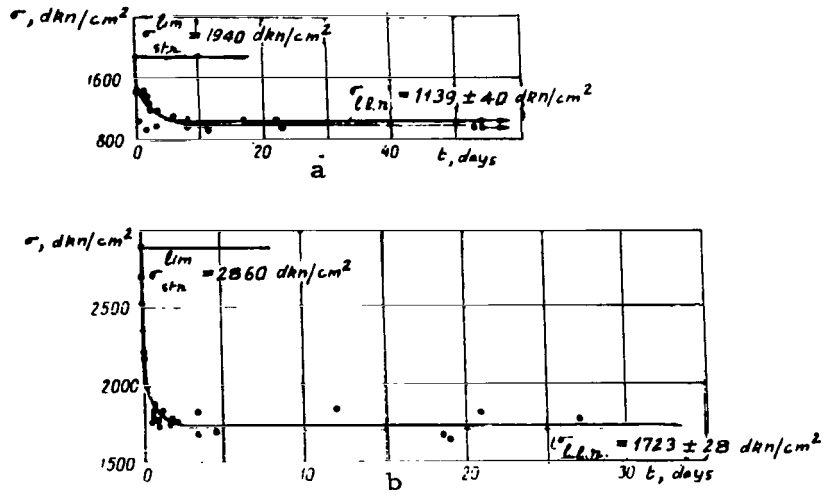

Fig. 6. Prolonged resistance curves of DSP-B under compression (a) and tension (b) along the ply layers along the fiber lining.

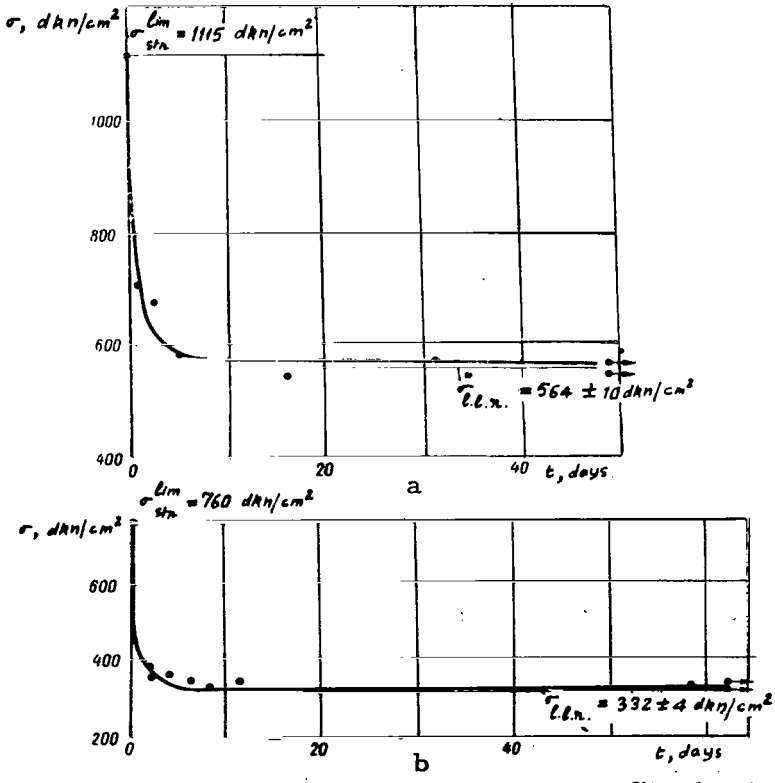

Fig. 7. Prolonged resistance curves of pine woods (W=10%) subjected to tension (a) and compression (b) along the fibers.

Table 4

Material	Direction of application of force	Limit of long-term resistance, dkn/cm^2	Time base t_2 in days	Coefficient C_1	$\frac{\sigma_{l.l.r.}}{\sigma_{str}}$, %
DSP-B	Compression along the ply layers along the fibers of lining	1140	2.6	0.90	59
	Compression along the ply layers perpendicular to the fibers of lining	735	1.1	0.56	59
	Compression perpendicular to the ply layers	735	3.5	0.02	50
	Tension along the ply layers along the fibers of lining	1723	1.2	0.45	60
DSP-V	Compression along the ply layers along the fibers of lining	750	0.3	0.62	51
	Compression perpendicular to the ply layers	1330	1.0	0.25	52
	Tension along the ply layers along the fibers of lining	744	5.7	0.35	50

Contd.

1	2	3	4	5	6
DSP-G	Compression along the ply layers along the fibers of lining	740	0.9	0.30	47
	Compression perpendicular to the ply layers	1230	0.3	0.68	48
	Tension along the ply layers along the fibers of lining	797	2.4	0.56	60
KAST-V	Compression along warp	617	1.5	0.69	82
	Compression along weft	536	0.4	0.28	74
	Tension along warp	1340	0.5	0.35	90
	Tension along weft	1000	2.0	0.79	76
VFT-S	Compression along warp	1070	2.6	0.85	62
	Compression along weft	738	1.5	0.63	58
SVAM	Compression along ply layers along fibers	1160	1.2	0.07	33
Pine	Compression along fibers	332	4.0	0.63	44
	Tension along fibers	564	2.5	0.42	52

Contd.

1	2	3	4	5	6
Beech	Compression along fibers	310	14.0	0.68	47
	Tension along fibers	863	5.0	0.48	44
Oak	Compression along fibers	430	1.8	0.28	60
	Tension along fibers	555	5.0	0.43	46

DETERMINATION OF INSTANTANEOUS MODULUS OF ELASTICITY

Instantaneous modulus of elasticity may be determined:

1) from short duration machine tests, as the tangent of the angle of inclination of the tangent passing through the origin of the diagram of deformations;

2) from tests on deformation, corresponding to a load at the moment of its application, under the action of a constant prolonged load;

3) from stepwise tests.

In the first case, particularly when the base of measurement is equal to the height of the specimen, alterations in the initial portion of the deformation diagram affect the value of the instantaneous modulus of elasticity. If the measurements are taken on a base less than the height of the specimen, the error is considerably decreased.

In the second case, the value of the instantaneous modulus of elasticity may be obtained more accurately than in the first case, but its value is somewhat underestimated due to the fact that the development of the deformation does not take place

instantaneously under the action of a constant load. Hence the obtained characteristics of deformability of the material is often called conditionally instantaneous.

In the third case, in measuring the deformations at every stage of loading at the moment of application of the load, it is possible to obtain from experiment several values of instantaneous modulus of elasticity for one and the same specimen and take the mean of the results. This is a more rational method than the previous one.

The third method was mainly used in the investigation of laminated plastics and wood. Instantaneous modulus of elasticity was determined at stresses, lower and higher than the limit of long-term resistance, i. e., in the elastic and elasto-plastic region of work of the material during loading as well as during unloading. As shown by experiments, in all these cases, the instantaneous modulus of elasticity follows the proportionality rule (Fig. 8). It should be noted that these values are underestimated, since deformations were measured on a base equal to the height of the specimen and deformation due to crumpling of the ends of the specimens in the course of loading is superimposed on the deformation due to compression.

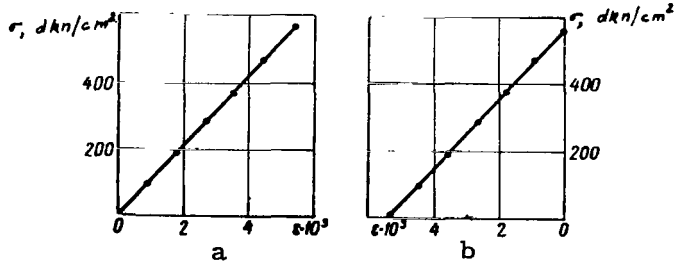

Fig. 8. Diagrams of instantaneous elastic deformations of KAST-V subjected to compression along the weft.
a - loading; b - unloading.

For eliminating the influence of crumpling on the value of the modulus of elasticity in the case of compression, deformations were measured by tensometers at a base less than the specimen height. The results of the experiment are presented in Table 5, where the values of instantaneous modulus of elasticity in the case of tension obtained from diagram of deformations are also presented. Measurement of tensile deformations

Table 5

Material	Direction of action of force	Instantaneous modulus of elasticity $E_{inst} \times 10^{-5}$, dkn/cm^2
DSP-B	Compression along the ply layers along the fibers of lining	3.40
	Compression along the ply layers transverse to the fibers of lining	0.74
	Compression perpendicular to the ply layers	0.39
	Tension along the ply layers along the fibers of lining	3.60
	Tension along the ply layers transverse to the fibers of lining	0.72
KAST-V	Compression along weft	2.22
	Compression along warp	2.74
	Tension along weft	2.40
	Tension along warp	2.95
VFT-S	Compression along warp	2.55
	Tension along warp	2.50
Pine	Compression along fibers	1.63
	Tension along fibers	1.76

were taken with the help of strain pickups or Gugenberger tensometers. As seen, the values of instantaneous modulus of elasticity in the case of compression and tension are practically equal.

DETERMINATION OF PROLONGED MODULUS OF ELASTICITY

Prolonged modulus of elasticity is a characteristic of the maximum deformability of a material in the elastic region of its work. Its value is determined from the creep curve at stresses less than the limit of the long-term resistance as a ratio of the given constantly acting stresses to the asymptotic value of the total deformation due to these stresses:

$$E_{prol} = \frac{\sigma}{\varepsilon_{total}} .$$

In the equation (1.27), the coefficients a, b, c and the time base t_1 are determined from the experimental data by processing them statistically. Coefficient α depends on the allowance for determining the deformations. The coefficient c, according to the boundary conditions of the creep curves lies within the limits $0 < c < 1$.

Since the value of instantaneous modulus of elasticity can be determined from stepwise tests, it is possible to isolate from the creep curve a part of deformation equal to the instantaneous deformation and to consider only the deformation increasing with time. In order to simplify the determination of prolonged modulus of elasticity from the creep curve, c may be approximately taken to be equal to one. This will distort the creep curve in the region of small time intervals but will simplify considerably the calculation of the prolonged modulus of elasticity.

For each curve showing the development of deformation, by substituting the function

$$\varepsilon_t = a\left(1 - e^{-qt}\right),$$

we obtain the coefficients a and q, by the method of least squares, in the equation of this curve, which deviates the least from the experimental data.

The coefficient a gives the asymptotic value of deformation increasing with time and q characterizes the slope of the curve and serves to determine the time base while keeping in view the allowance on the total deformation. The latter is equal to the sum of the instantaneous elastic deformation and the deformation developing with time.

The time base t_1 for the creep curve is obtained from the equation of total deformations

$$\varepsilon_{total} = \varepsilon_{inst} + a\left(1 - e^{-qt}\right).$$

If the allowance is equal to 5% of the total deformation and time of development of deformations $t = t_1$ the value of the time base t_1 may be obtained from the equation

$$0.95\,\varepsilon_{total} = \varepsilon_{inst} + a\left(1 - e^{-qt}\right),$$

and hence

$$t_1 = -\frac{\ln\left(1 - \dfrac{0.95\,\varepsilon_{total} - \varepsilon_{inst}}{a}\right)}{q},$$

where

$$\varepsilon_{total} = \frac{\sigma}{E_{prol}}, \quad \varepsilon_{inst} = \frac{\sigma}{E_{inst}}, \quad a = \frac{\sigma}{E_{prol}} - \frac{\sigma}{E_{inst}};$$

q is the coefficient obtained experimentally and having the dimension 1 day, E_{prol} is the prolonged modulus of elasticity, E_{inst} is the instantaneous modulus of elasticity and σ is the constantly acting stress.

The determination of the coefficient c in the equation (1.27) by the method of least squares, when the values of instantaneous and prolonged moduli of elasticity are known, is analogous to the method mentioned earlier for determining the coefficient c_1 in the equation of the prolonged resistance curve.

Fig. 9 shows the experimentally obtained creep curves (points) and the curve plotted according to the equation (1.27). As is obvious from the figure, theoretical and experimental

values of deformations fairly coincide with each other. Analogous creep curves were obtained by other investigators [10, 20, 21, 38, 53] as well.

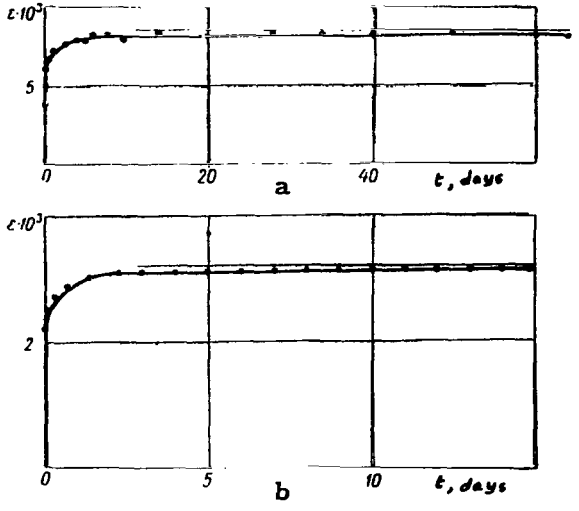

Fig. 9. Creep curves in the elastic region of work of a material. a - compression of KAST-V along weft (σ = 450 dkn/cm²); b - tension of VFT-S along warp (σ = 560 dkn/cm²).

Table 6 represents the characteristics of deformability of laminated plastics subjected to compression. Since deformations were measured on the base, which was equal to the height of the specimen, the value of the modulus of elasticity is underestimated due to the influence of crumpling at the ends of the specimens.

Table 7 represents some data on the characteristics of deformability of laminated plastics and wood obtained from the compression creep curve, when the base of measurement of deformations is less than the height of the specimens, and from the tensile creep curves. As seen, the value of the prolonged modulus of elasticity in the case of compression and tension may be considered to be practically equal.

Table 6

Material	Direction of action of force	Instantaneous modulus of elasticity $E_{inst} \times 10^{-5}$, dkn/cm^2	Prolonged modulus of elasticity $E_{prol} \times 10^{-5}$, dkn/cm^2	Time base t_1 in days	c	$\frac{E_{prol}}{E_{inst}}$, %
DSP-B	Along the ply layers along the fibers of lining	1.86	0.535	4.0	0.309	29
	Along the ply layers across the fibers of lining	0.46	0.15	7.2	0.224	33
	Perpendicular to the ply layers	0.43	0.18	4.0	0.186	42
DSP-V	Along the ply layers along the fibers of lining	1.06	0.26	6.0	0.313	25
	Perpendicular to the ply layers	0.53	0.23	9.0	0.175	43
DSP-G	Along the ply layers along the fibers of lining	0.80	0.25	5.6	0.253	31
	Perpendicular to the ply layers	0.57	0.21	5.5	0.238	37

Contd.

1	2	3	4	5	6	7
KAST-V	Along warp	1.34	0.73	4.5	0.230	55
	Along weft	1.13	0.53	3.8	0.250	47
VFT-S	Along warp	1.67	0.92	6.2	0.184	55
	Along weft	1.34	0.69	4.8	0.240	51
SVAM	Along the ply layers	3.60	-	-	-	-

Table 7

Material	Direction of action of force	Prolonged modulus of elasticity $E_{prol} \times 10^{-5}$, dkn/cm2	Time base t_1 in days	c	$\frac{E_{prol}}{E_{inst}}$, %
DSP-B	Compression along the ply layers along the fibers of lining	2.80	4.2	0.322	92
	Compression along the ply layers across the fibers of lining	0.41	6.8	0.205	55
	Compression perpendicular to the ply layers	0.15	5.5	0.170	38
	Tension along the ply layers along the fibers of lining	2.38	4.6	0.294	52

Contd.

1	2	3	4	5	6
KAST-V	Compression along warp	1.59	4.8	0.238	58
	Tension along warp	1.80	5.2	0.296	61
	Compression along warp	1.60	6.2	0.235	63
	Tension along warp	1.75	4.7	0.236	70
VFT-S	Compression along fibers	0.78	11.0	0.132	48
Pine	Tension along fibers	0.80	10.0	0.144	45

INVESTIGATION OF LAW OF PROPORTIONALITY WITH TIME

It has been stated earlier that there exists a law of proportionality between stresses and instantaneous deformations. Some works [10, 13, 20] mention that the law of proportionality is also valid for elasto-viscous deformations. To verify this, experiments were conducted on creep in the elastic region of work of the material at different values of constantly acting stresses. The curve of dependence of the ratio of elasto-viscous deformations to elastic instantaneous deformations upon the time of action of the constant load was plotted on the basis of the experimental results.

Some results verifying the validity of the law of proportionality for laminated plastics and wood subjected to compression and tension are presented in Figs. 10 and 11. As seen, the interval between the creep curves at different loads is very small. This permits us to say that at every moment the value of the total elasto-viscous deformation is proportional to the value of the constant load. Hence, on the basis of the proportion-

ality law, it can be concluded that time bases t_1, coefficients c and the prolonged moduli of elasticity E_{prol} are constant and do not depend upon the value of the acting stresses.

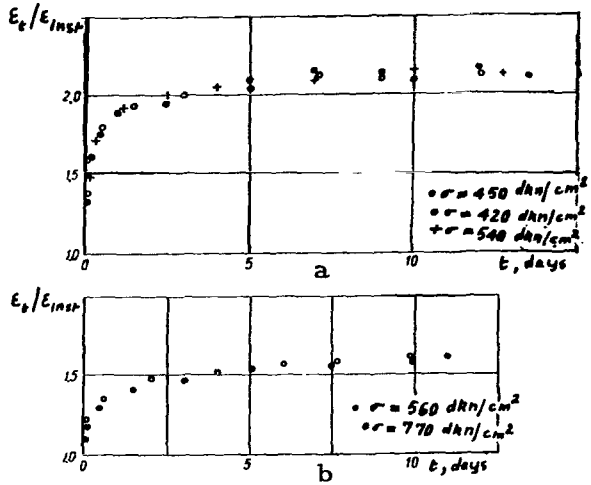

Fig. 10. Curves showing the development of the ratio of the total elastic and elastic instantaneous deformations with time.

a - compression of KAST-V along weft; b - tension of VFT-S along warp.

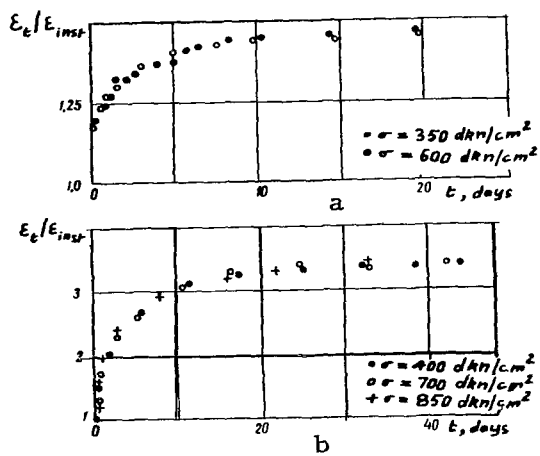

Fig. 11. Curve showing the development of the ratio of total elastic and elastic instantaneous deformations with time.

a - tension of oak wood along the fibers of lining; b - compression of DSP-G plastic, perpendicular to the layer of the ply.

Knowing the equation of creep curve, it is possible to obtain the law of change of modulus of elasticity with time. The latter changes with time from instantaneous to prolonged according to an asymptotic law. Since creep tests are generally conducted in the course of a restricted time interval of observations and prolonged modulus is found by extrapolating the creep curve to an infinitely long time, the obtained value of the prolonged modulus of elasticity should approach its theoretical value to some extent. Hence it would be expedient to call this modulus as the restricted prolonged modulus of elasticity. In future, we shall refer, for simplicity, to the experimentally obtained restricted values of prolonged modulus of elasticity as the prolonged modulus of elasticity. As an illustration, curves of the change of moduli of elasticity of laminated plastics with time in the case of compression and tension by a constant load are presented in Fig. 12.

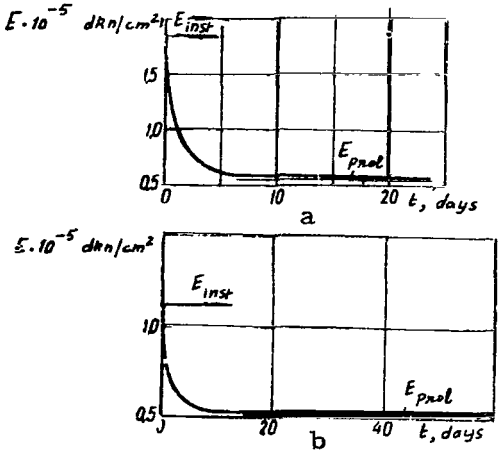

Fig. 12. Change of modulus of elasticity with time in the case of compression by a constant load.
a - DSP-B along the fiber of the lining along the ply; b - KAST-V along the weft.

REDEFORMATION AFTER UNLOADING IN THE ELASTIC REGION OF WORK OF MATERIAL

It has been stated earlier that under the action of a load, giving rise to stresses less than the limit of long-term resistance, elastic deformations are developed. It is shown that instantaneous elastic deformations get redeformed instantane-

ously on unloading. Let us consider the redeformation after loading, giving rise to the development of elasto-viscous deformations in the course of time interval T.

As shown by the experiments [13, 20, 56], in this case, during unloading, there appear instantaneous redeformation and redeformation developing with time. Fig. 13 presents some creep diagrams during loading and unloading, from which it is seen that redeformation of the total deformation attained in loading practically ends in the course of redeformation time interval equal to the deformation time interval.

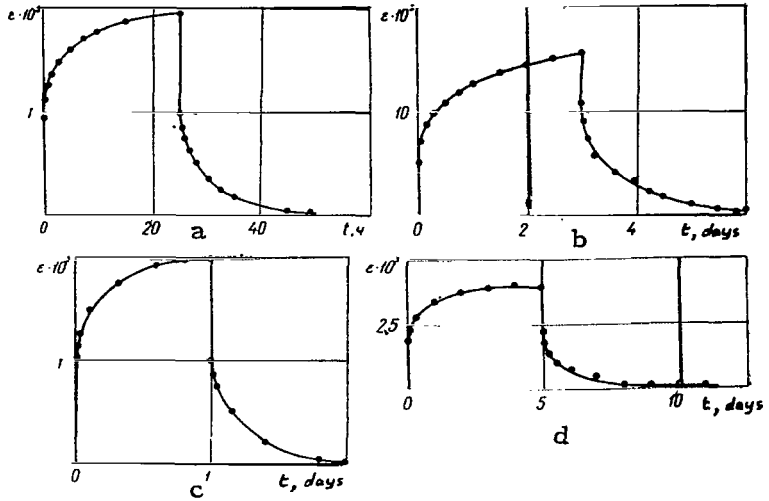

Fig. 13. Diagram of deformations and redeformation in the elastic region of work of material.

a - compression of pine wood along fibers (σ = 170 dkn/cm^2); b - compression of DSP-B across the fibers of lining along the ply layers (σ = 240 dkn/cm^2); c - tension of VFT-S along warp (σ = 250 dkn/cm^2); d - compression of KAST-V along weft (σ = 220 dnk/cm^2).

Some residual redeformation after the expiry of the time interval appears as a result of the unavoidable errors in deformation measurements. Moreover, as the calculations show, only the external loading was eliminated on unloading, but the weight of the specimen itself caused deformation, which on accumulation gave rise to further residual deformation. But for explaining the elastic work of a material, the residual deformation of the experiment should not be considered as irreversible.

Thus, the conducted experiments showed, firstly, that deformations at stresses less than the limit of long-term resistance are elastic, and secondly, that redeformation law is the same as the deformation law. Deformation and redeformation process, in this case, consists of two periods: the period of formation of instantaneous elastic deformation or redeformation and the period of formation of elasto-viscous deformation or redeformation increasing with time at a diminishing rate.

CREEP AT STRESSES HIGHER THAN THE LIMIT OF LONG-TERM RESISTANCE

If a load is applied on material which gives rise to stresses higher than the limit of long-term resistance, the creep process in the course of some interval of time, depending upon the magnitude of the applied load, will lead to the failure of the material. Creep curves at stresses higher than the limit of long-term resistance are shown in Fig. 14. Analogous curves have also been obtained by other investigators [4, 10, 20, 38, 56].

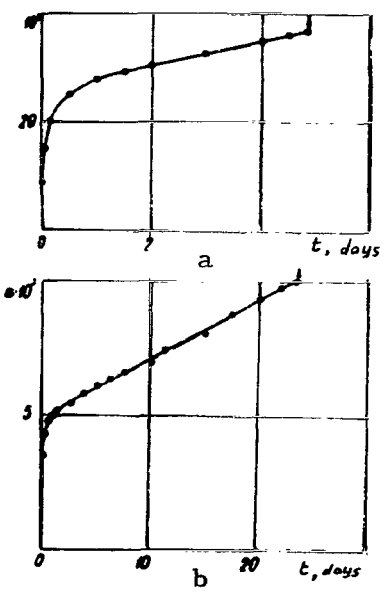

Fig. 14. Creep curves in the elastic and plastic regions of work of material. a - compression of DSP-B along the fibers of lining along the ply layers (σ = 1500 dkn/cm^2); b - tension of VFT-S along warp (σ = 800 dkn/cm^2).

Thus, it may be considered on the basis of the experimental data that if the constant stresses are higher than the limit of long-term resistance, deformations take place which increase in the course of a limited time interval, leading to the failure of the material at the end of this time interval. The deformation — time curve in the above case, as is known, breaks up into four parts with respect to time.

In part I, elastic instantaneous deformation takes place.

Elasto-viscous deformations appear in Part II, which develop with time at a decreasing rate of development of deformations. The growth of elasto-viscous deformations, developing in Part II, begins at a rate equal to the rate of development of instantaneous elastic deformations in Part I and continues with time at a gradually decreasing rate of development of deformations.

The growth of elasto-viscous deformations does not continue in the course of infinitely long time, but takes place during a limited time interval after the expiry of which the development of these deformations ceases and plastic deformations start developing with time. During the course of the stated time interval of development of elastic deformations, elastic deformations, at any given value of stresses, attain the maximum value.

Total elastic deformations (the sum of instantaneous elastic and elasto-viscous deformations) at the end of the time interval of development of elastic deformations attain a maximum value at the given stresses and may be termed as total limiting elastic deformation. The minimum rate of increase in elastic deformation, in this case, is also called as the limiting rate.

Since, at any given value of stresses, higher than the limit of long-term resistance, after the expiry of the time interval of development of elastic deformations, further rise in elastic deformations stops and the formation of plastic deformations begins, the stresses are the elastic limit at a constantly acting load.

At the end of the time interval of development of elastic deformations, when further growth of elastic deformation ceases, development of local plastic deformations begins, provided the values of the attained limiting elastic deformation remains constant.

If unloading is done at a moment corresponding to the total development of elastic deformations and partial development of plastic deformations, the redeformation process takes place as follows. After removing the load, at first the instantaneous redeformation takes place and this is equal in magnitude to the instantaneous deformation at the time of loading. Then, there takes place redeformation developing with time, which is equal in magnitude to the elastic deformation also developing with time. The duration of redeformation is equal to the duration of development of limiting elastic deformation. As a result, there remains the residual plastic deformation on unloading which is irreversible for any time interval and is equal in magnitude to the plastic deformation at the time of loading. Creep diagrams of loading and unloading in the case, when the applied stresses are higher than the limit of long-term resistance, are shown in Fig. 15.

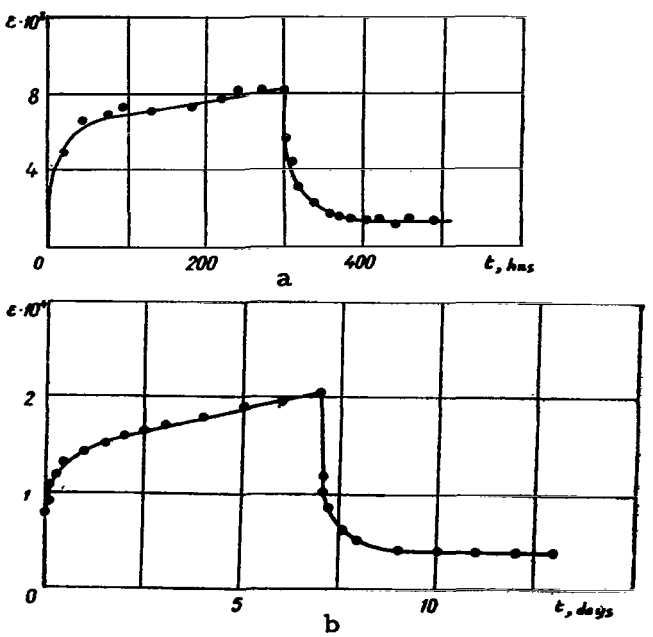

Fig. 15. Diagrams of deformations and redeformations in the elastic and plastic regions of work of material in the case of compression of pine wood along the fibers (a) (σ = 310 dkn/cm^2) and KAST-V along weft (b) (σ = 880 dkn/cm^2).

RELATION BETWEEN THE MAGNITUDE OF CONSTANT STRESSES, RATE OF DEVELOPMENT OF PLASTIC DEFORMATION AND LIMITING ELASTIC DEFORMATION

In the case of prolonged action of a load, this relation is established from the experiments on creep. For this purpose, creep diagrams are obtained at several stresses higher than the limit of long-term resistance. The rate of development of plastic deformations with time is determined on the linear part of the diagram in the period III, and the limiting deformation at the point at which the curved part of the diagram in the period II meets the linear part in the period III.

Figs. 16 and 17 show the relation between the stresses $\sigma > \sigma_{l.l.n}$, the rate of plastic deformation v_{plast} and the limiting elastic deformation ε_s. As seen, the relation between σ and v_{plast} has a nonlinear asymptotic nature, whereas the relation between σ and ε_s is linear. Points corresponding to the critical deformations at $\sigma \to \sigma_{l.l.n}$ and $\sigma \to \sigma_{str}^{lim}$, and to the values calculated from the moduli of elasticity mentioned earlier are plotted in the left hand side of the curves. These points are well in line with the linear relation between σ and ε_s. Hence, this linear relation is justified at all stresses ranging from $\sigma_{l.l.n.}$ to σ_{str}^{lim}.

The linear relation between σ and v_{plast} is often used in literature [20, 38, 83, 89]. However, it should be mentioned, that this linear relation was verified experimentally in a comparatively narrow range of stresses, mainly, lower ones (1.1-1.3 $\sigma_{l.l.n.}$). In Figs. 16 and 17, the linear relation between σ and v_{plast} for groups of experimental points corresponding to the stresses σ = 1.1-1.4 $\sigma_{l.l.n.}$ may be selected from the right hand side. As seen from the curves, the relation between σ and v_{plast} may be considered with sufficient accuracy as linear in the said range of stresses.

Thus, on the basis of the study of large experimental data on creep and prolonged strength of plastic glasses, DSP-laminated plastics and wood, it is possible to establish for this class of elasto-visco-plastic materials principal constants, characteriz-

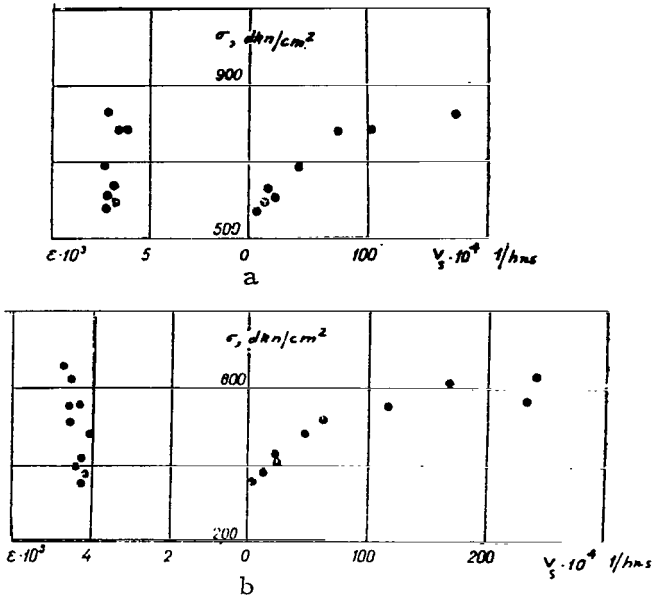

Fig. 16.Relation between the stresses, the rate of plastic deformation and the limiting elastic deformation for pine wood subjected to tension (a) and compression (b) along the fibers.

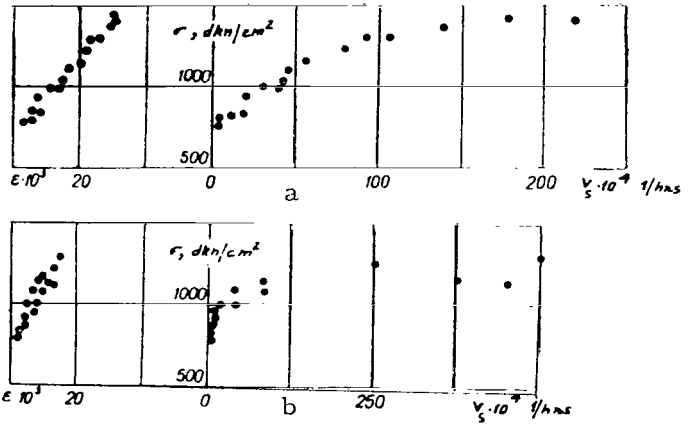

Fig. 17.Relation between the stresses, the rate of plastic deformation and the limiting deformation in the case of DSP-V (a) and DSP-G (b) subjected to compression along the ply layers along the fibers of lining.

ing the variation in strength and their deformability with time. Following are these constants:

$\sigma_{str.}^{lim}$ –limit of ultimate strength;

$\sigma_{l.l.r.}$–limit of long-term resistance;

$E_{inst.}$ –instantaneous modulus of elasticity;

$E_{prol.}$ –prolonged modulus of elasticity;

t_1 - time base for creep curve;

t_2 - time base for prolonged resistance curve;

c and c_1 –coefficients, characterizing the slope of the creep curve and the prolonged resistance curve.

All these constants, as stated earlier, may be obtained from experiments by the method of statistical analysis of creep curves and prolonged resistance curves of known functions.

CHAPTER III

RESISTIBILITY AND DEFORMABILITY AT PROLONGED ACTION OF A CONSTANT LOAD

LAWS OF DEVELOPMENT OF DEFORMATIONS IN ELASTIC REGION OF WORK OF MATERIAL

Let us consider the laws of development of deformations in the elastic region of work of a material, i.e., when a constant load gives rise to stresses less than the limit of long-term resistance.

It is known from experiments that the action of a constant load in the course of a given time interval gives rise to instantaneous elastic and elasto-viscous deformations. When the time interval of development of deformations $t \to 0$, total deformation, corresponding to the given constantly acting stresses σ, is

$$\varepsilon_{total} \longrightarrow \varepsilon_{inst} = \frac{\sigma}{E_{inst}}. \tag{3.1}$$

In the case of a given finite time interval of action of stresses $t = T$

$$\varepsilon_{total} = \varepsilon_{inst} + \varepsilon_t = \frac{\sigma}{E_t}. \tag{3.2}$$

In the case of infinite large time interval of action of stresses

$$\varepsilon_{total} \longrightarrow \varepsilon_{prol} = \frac{\sigma}{E_{prol}}. \tag{3.3}$$

In the formulas (3.1) - (3.3), E_{inst} is the instantaneous modulus of elasticity; E_{prol} is the prolonged modulus of elasticity; and E_t is the modulus of elasticity varying with time.

As mentioned earlier, the boundary conditions and the experimental data are well satisfied by an exponential function for the creep curve

$$\varepsilon_{total} = a + be^{-\alpha\left(\frac{t}{t_1}\right)^c}, \tag{3.4}$$

where a and b are constants depending on the characteristics of deformability of the material; c is a constant characterizing the rate of development of elasto-viscous deformations with time; t_1 is the time base of test, at which the total elastic deformation developing with time attains a value differing by L% from the magnitude of total elastic deformation at infinitely long time of action of load and α is the constant depending upon the allowance (%) in deformation while determining the time base of test.

It is also known from experiments that the rate of growth of elastic deformations with time under the action of a constant load changes within the limits

from

$$V = \frac{d\varepsilon_{total}}{dt} \longrightarrow \infty \quad \text{at } t \longrightarrow 0, \tag{3.5}$$

to

$$V \longrightarrow 0 \qquad \text{at } t \longrightarrow \infty. \tag{3.6}$$

The function (3.4) satisfies the given limiting values of the rate of elastic deformation with time. In fact

$$V = \frac{d\varepsilon_{total}}{dt} = -\frac{ba c t^{c-1} e^{-\alpha\left(\frac{t}{t_1}\right)^c}}{t_1^c}. \tag{3.7}$$

On the basis of the conditions (3.5) and (3.6), the value of the coefficient c in the equations (3.4) and (3.7) must be within the limits $0 < c < 1$. From the boundary conditions (1.24), let us determine the coefficients a and b:

$$a + b = \varepsilon_{inst},$$

$$a = \varepsilon_{prol},$$

from which

$$b = \varepsilon_{inst} - \varepsilon_{prol}.$$

The coefficient α depends on the magnitude of the allowance for determining the deformation, time of whose development is t_1.

If the allowance on the deformation is L% of ε_{prol}, by putting the values of a, b and $t = t_1$ in (3.1), we get

$$\left(1 - \frac{L}{100}\right)\varepsilon_{prol} = \varepsilon_{prol} + (\varepsilon_{inst} - \varepsilon_{prol})\, e^{-\alpha},$$

where

$$\alpha = -\ln \frac{L\,\varepsilon_{prol}}{100(\varepsilon_{prol} - \varepsilon_{inst})} \tag{3.8}$$

From the expression (3.8), it is obvious that the coefficient α in the case of a constant load depends on the value of the allowance L and on the elastic characteristics of deformability of material.

Putting the obtained values of the coefficients a and b in the equation (3.4), we get

$$\varepsilon_{total} = \varepsilon_{prol}\left[1 + \left(\frac{E_{inst}}{E_{prol}} - 1\right) e^{-\alpha\left(\frac{t}{t_1}\right)^c}\right] \tag{3.9}$$

Putting the values of ε_{inst} and ε_{prol}, obtained in the case of a constant stress σ, in the expression $\frac{\varepsilon_{inst}}{\varepsilon_{prol}} - 1$ in the equation (3.9), we get

$$\frac{\varepsilon_{inst}}{\varepsilon_{prol}} - 1 = \frac{E_{prol}}{E_{inst}} - 1.$$

Let us write k for $1-\frac{E_{prol}}{E_{inst}}$ and substitute in the equation (3.9) ε_{prol} by $\frac{\sigma}{E_{prol}}$ both being equal. Then the final expression for the total elastic deformation, developing with time, will have the form

$$\varepsilon_{total} = \frac{\sigma}{E_{prol}}\left[1-ke^{-\alpha\left(\frac{t}{t_1}\right)^c}\right] \qquad (3.10)$$

It follows from the expression (3.10) that, with a change in the time interval t, the magnitude of the total elastic deformations changes at a constant value of the given stresses, i.e., the elastic characteristic of deformability of the material changes with time. This elastic characteristic of deformability of material changing with time is called the modulus of elasticity variable with time.

E_t, the modulus of elasticity variable with time, may be obtained from the expression for total deformation

$$E_t = \frac{E_{prol}}{1-ke^{-\alpha\left(\frac{t}{t_1}\right)^c}}, \qquad (3.11)$$

or by substituting E_{prol} by $(1-k)\,E_{inst}$.

$$E_t = \frac{(1-k)E_{inst}}{1-e^{-\alpha\left(\frac{t}{t_1}\right)^c}} \qquad (3.12)$$

If unloading is done after the expiry of a definite time interval T, i.e., if the acting load is removed, the instantaneous redeformation takes place under the influence of internal elastic forces, which is equal in magnitude to the instantaneous elastic deformation and afterwards, there sets in elastic (viscous) redeformation, which continues with time. The law of change in this redeformation with time is the same as the law of change in elastic (viscous) deformation with time.

LAWS OF DEVELOPMENT OF DEFORMATIONS IN PLASTIC REGION OF WORK OF MATERIAL

Let us assume that a constantly acting load gives rise to stresses greater than the limit of long-term resistance. The process of deformation with time, in this case, as has been stated earlier, is divided into four periods.

Instantaneous elastic deformation, proportional to the value of the acting stresses, arises in the first period; elastic (viscous) deformation develops with time in the second period, and at every time interval, its magnitude is related to the stresses by the proportionality law. In the third period, development of plastic deformation with time at a constant rate takes place whereas in the fourth period there occurs the process of fracture of the material connected with complete separation of its parts and proceeding at a high speed.

The values of the limiting elastic deformations, corresponding to the rates of development of deformations, stresses and time intervals may be obtained from the relations stated earlier by putting the values $\varepsilon = \varepsilon_s$, $\sigma = \sigma_s$, $t = t_s$, $v = v_s$.

The total limiting elastic deformation is given by the expression

$$\varepsilon_s = \frac{\sigma_s}{E_{prol}} \left[1 - Re^{-\alpha \left(\frac{t_s}{t_1} \right)^c} \right] \tag{3.13}$$

The rate of development of the limiting elastic deformations is expressed as

$$v_s = \frac{d\varepsilon_s}{dt_s} = \frac{\sigma_s R \alpha t_s^{c-1} c}{E_{prol} t_1^c} e^{-\alpha \left(\frac{t_s}{t_1} \right)^c} \tag{3.14}$$

The total limiting elastic deformation may be expressed, depending upon the rate v_s, in the form

$$\varepsilon_s = \frac{\sigma_s}{E_{prol}} \left[1 - \frac{v_s E_{prol} t_1^c}{\sigma_s \alpha t_s^{c-1} c} \right] \tag{3.15}$$

The development of plastic deformation with time, at a constant rate equal to the rate of development of the limiting elastic deformation, starts under the action of constant stresses after the expiry of the time interval t_s. With the change in the magnitude of stresses, the magnitude of the corresponding rate of development of limiting elastic deformation changes, and hence also the rate of development of plastic deformations.

In order to establish the deformation rules in the plastic region of work of material at constant and prolonged acting loads, it is essential to know the relation between elastic limit σ_s and limiting elastic deformation ε_s which demarcates the elastic and plastic regions of work of material. Repeated experiments by us as well as the investigations by V.N. Bykovskiĭ [18 - 20] and A.M. Ivanov [35, 36] showed that a linear relation exists between stresses σ_s and deformation ε_s which, in a general form, may be written as

$$\sigma_s = a + b\varepsilon_s \,, \tag{3.16}$$

where a and b are constants determining the limits of application of the said linear relation.

The coefficients a and b may be obtained from limiting values of σ_s and ε_s and depend on the characteristics of instantaneous and prolonged strength and deformability of the material under investigation. Following are the conditions for obtaining the coefficients a and b:

when

$$t \to 0 \text{ and } \sigma_s \to \sigma_{str}^{lim} \qquad \varepsilon_s \to \varepsilon_{inst} = \frac{\sigma_{str}^{lim}}{E_{inst}} \,,$$

when

$$t \to \infty \text{ and } \sigma_s \to \sigma_{l.l.n.} \qquad \varepsilon_s \to \varepsilon_{prol} = \frac{\sigma_{l.l.n.}}{E_{prol}} \,. \tag{3.17}$$

Using the equation (3.16) and conditions (3.17), we get

$$a = \sigma_{l.l.n.} - \frac{\sigma_{l.l.n.}(\sigma_{str}^{lim} - \sigma_{l.l.n})}{E_{prol}\left(\frac{\sigma_{str}^{lim}}{E_{inst}} - \frac{\sigma_{l.l.n.}}{E_{prol}}\right)}, \quad b = \frac{\sigma_{str}^{lim} - \sigma_{l.l.n}}{\frac{\sigma_{str}^{lim}}{E_{inst}} - \frac{\sigma_{l.l.n}}{E_{prol}}}.$$

Let us write $\frac{\sigma_{l.l.n.}}{\sigma_{str}^{lim}} = k_2$ and $\frac{E_{prol}}{E_{inst}} = 1-k$ after putting the values of the coefficients a and b in the equation (3.16). We then obtain the relation between stresses σ_s and deformation ε_s as

$$\sigma_s = \frac{(1-k_2)E_{prol}\,\varepsilon_s - k\sigma_{l.l.n.}}{1-k-k_2} \tag{3.18}$$

or

$$\varepsilon_s = \frac{k(\sigma_{l.l.n.} - \sigma_s)}{(1-k_2)E_{prol}} + \frac{\sigma_s}{E_{prol}} \tag{3.19}$$

Solving the equations (3.13) and (3.19) simultaneously, we get the relation between elastic limit σ_s and time interval t_s as follows

$$\sigma_s = \frac{\sigma_{l.l.n.}}{1-(1-k_2)\,e^{-\alpha\left(\frac{t_s}{t_1}\right)^c}} \tag{3.20}$$

The equation (3.20) gives the relation between the change in elastic limit and time and satisfies the boundary conditions:

when

$$t_s \to 0 \qquad \sigma_s \to \sigma_{str}^{lim},$$

when

$$t_s \to \infty \qquad \sigma_s \to \sigma_{l.l.n} \tag{3.21}$$

Let us find the relation between the stresses σ_s and the rate of development of the limiting elastic deformations v_s. Solving the equations (3.14) and (3.20) simultaneously, we get

$$V_s = \frac{\sigma_s R \left(1 - \frac{\sigma_{e.l.n.}}{\sigma_s}\right) \alpha^{\frac{1}{c}} C}{E_{prol} (1 - R_2) t_1} \left(\ln \frac{1 - R_2}{1 - \frac{\sigma_{e.l.n.}}{\sigma_s}} \right)^{1 - \frac{1}{C}} . \quad (3.22)$$

When the stresses $\sigma \to \sigma_{e.l.n.}$ the rate $v_s \longrightarrow 0$. The rate v_s increases with an increase in stresses σ_s and when $\sigma_s \to \sigma_{str}^{lim}$, $V_s \to \infty$, which in the first gase gives value of the limiting elastic deformation $\varepsilon_s \longrightarrow \frac{\sigma_{e.l.n.}}{E_{prol}}$ and in the second case $\varepsilon_s \to \frac{\sigma_{str}^{lim}}{E_{inst}}$. The value of plastic deformation in both the cases is equal to zero, i. e., a brittle fracture occurs without the formation of plastic deformations.

Graphically the equation (3.22) can be represented as an asymptotic curve which has already been pointed out in experiments on laminated plastics and wood (see Ch. II).

Using the equations (3.13) and (3.20), it is possible to get the relation between limiting elastic deformation ε_s and the time interval of its development t_s

$$\varepsilon_s = \frac{\sigma_{e.l.n.} \left[1 - Re^{-\alpha \left(\frac{t_s}{t_1} \right)^c} \right]}{E_{prol} \left[1 - (1 - R_2) e^{-\alpha \left(\frac{t_s}{t_1} \right)^c} \right]} . \quad (3.23)$$

The value of total deformation in the third period consists of the limiting elastic deformation ε_s and the plastic deformation ε_{plast}, which under the action of the given stresses σ_s, develops in the course of the given time interval at a rate v_s:

$$\varepsilon_{total} = \varepsilon_s + \varepsilon_{Plast} \quad (3.24)$$

The value of plastic deformation is

$$\varepsilon_{Plast} = v_s (t - t_s), \quad (3.25)$$

where t is the time interval of action of the load; t_s is the time interval of development of the limiting plastic deformation; and v_s is the rate of development of the limiting elastic deformation.

Putting the value of ε_s and v_s in (3.24) and after transformation, we get

$$\varepsilon_{total} = \frac{\sigma_s}{E_{prol}} \left\{ 1 - ke^{-\alpha\left(\frac{t_s}{t_1}\right)^c \left[1 - \alpha\left(\frac{t_s}{t_1}\right)^c \left(\frac{t}{t_s} - 1\right)\right]} \right\} \quad (3.26)$$

Since the process of increase of separation of the elements of the macrostructure of the material takes place in the stage IV of development of deformation, investigations of this process do not give any additional data in the study of the formation and development of deformations. Usually, this period is neglected in the investigation of deformability of material as a whole, all the more so, as with time this stage comes to an end quickly. The formula (3.26) permits the determination of total value of deformation, after which the failure takes place. In order to determine the total deformation after the development of which failure takes place at a given stress, it is essential to know the value of time interval of the total development of deformation. Since the long-term resistance curve represents the diversity of ultimate strength of the material depending upon the time interval till failure t_{fr}, by knowing its equation it is possible to obtain the value of the time interval t_{fr} for the given stresses σ_s.

The equation of the long-term resistance curve, as stated earlier, may be expressed in terms of an exponential function (1.23).

Substituting the ratio $\frac{\sigma_{l.l.r.}}{\sigma_{str}^{lim}}$ by k_2 in the equation (1.23), we get the equation of the long-term resistance curve in the following form

$$\sigma_s = \sigma_{l.l.r.} \left[1 + \frac{1 - k_2}{k_2} e^{-\alpha_1\left(\frac{t_{fr}}{t_2}\right)^{c_1}} \right] . \quad (3.27)$$

Under the condition that at $t_{fr} = t_2$, the value of σ_s is greater than $\sigma_{l.l.r.}$ by L%, the value of α_1 is

$$\alpha_1 = -\ln \frac{L k_2}{100(1-k_2)} . \qquad (3.28)$$

By equating the expression for stresses σ_s according to the formulas (3.20) and (3.27), we get the relation between the time interval till failure t_{fr} and time interval t_s:

$$t_{fr} = t_2 \left[-\frac{1}{\alpha_1} \ln \frac{k_2}{e^{\alpha \left(\frac{t_s}{t_1}\right)^c} - (1-k_2)} \right]^{\frac{1}{c_1}} \qquad (3.29)$$

In order to obtain the total deformation at the moment of failure, depending on the value of the stresses σ_s, it is necessary to put the time interval $t = t_{fr}$ in (3.26) according to the formula (3.29) and substitute the time interval t_s, depending on σ_s, by the formula (3.20). After substitution and transformation, we get

$$\varepsilon_{fr} = \frac{\sigma_s}{E_{prol}} \left| 1 - \frac{k}{1-k_2} \left(1 - \frac{\sigma_{l.l.r.}}{\sigma_s}\right) \times \right.$$

$$\times \left\{ 1 + \ln \frac{1 - \frac{\sigma_{l.l.r.}}{\sigma_s}}{1-k_2} \left[\frac{t_2 \left\{ -\frac{1}{\alpha_1} \ln \frac{k_2 \left(\frac{\sigma_s}{\sigma_{l.l.r.}} - 1\right)}{1-k_2} \right\}^{\frac{1}{c_1}}}{t_1 \left\{ -\frac{1}{\alpha_1} \ln \frac{1 - \frac{\sigma_{l.l.r.}}{\sigma_s}}{1-k_2} \right\}^{\frac{1}{c}}} \right] \right\} \left. \right| . \qquad (3.30)$$

If the value of σ_s, depending on the time interval till failure t_{fr}, is put in the equation (3.30), the expression for de-

termining the total deformation at failure ε_{fr} can be obtained in the form:

$$\varepsilon_{fr} = \frac{\sigma_{str}^{lim}}{E_{prel}} \left| R_2 + e^{-\alpha_1 \left(\frac{t_{fr}}{t_2}\right)^{c_1}} \times \right.$$

$$\times \left\{ 1 - R_2 - R \left[1 + \left(\frac{t_{fr}}{t_1 \sqrt[c]{\frac{1}{\alpha} \ln \frac{1}{R_2 e^{\alpha_1 \left(\frac{t_{fr}}{t_1}\right)^{c_1}} + 1 - R_2}}} - 1 \right) \times \right.\right.$$

$$\left.\left. \times \ln \frac{1}{1 - R_2 + R_2 e^{\alpha_1 \left(\frac{t_{fr}}{t_1}\right)^{c_1}}} \right] \right\} \Bigg| . \qquad (3.31)$$

The expression (3.31) gives the equation of the curve, enveloping all curves of deformations which develop with time in the case of prolonged acting loads.

Utilizing the rules of the development of deformations with time and change in resistibility presented above, it is possible to solve a number of other problems, such as establishing the laws of change in resistibility with time in the case of a constant given initial deformation (relaxation process), law of development of deformations and change in resistibility in stepwise loading and deforming, rules of development of deformations and change in resistibility in tests with a constant loading or deforming rate, etc.

As an example, curves of development of deformations with time at constantly acting load of different magnitudes in the case of compression, obtained by using the abovementioned formulas, are shown in Fig. 18.

Figs. 19 and 20 show the relations between stresses and deformations caused by the action of a constant load in the case of compression of the DSP-plastic, plastic glass and wood.

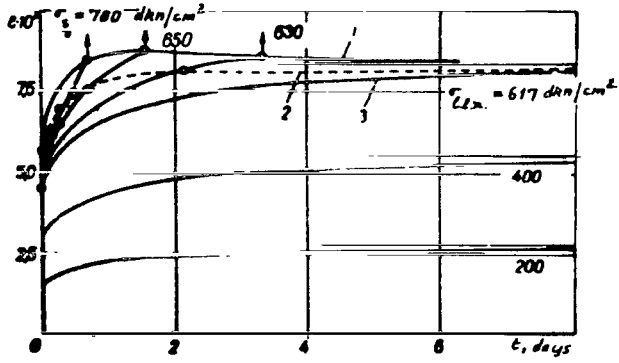

Fig. 18. Curves of deformations at constant loads in the case of compression of KAST-V along warp.

1 - Curve of deformations at failure; 2 - Curve of limiting elastic deformations; 3 - Limiting curve of limiting deformations.

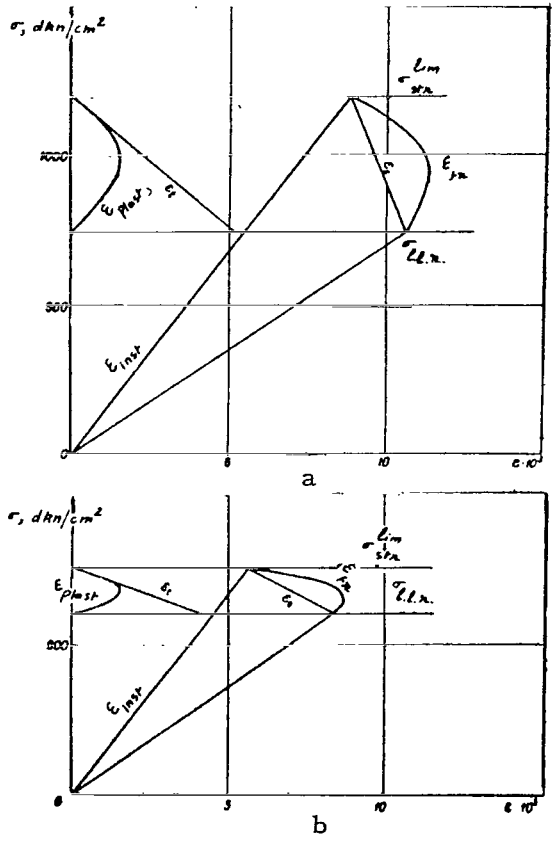

Fig. 19. Relation between stresses and deformations in the case of prolonged compression of VFT-S along weft (a) and KAST-V along warp (b).

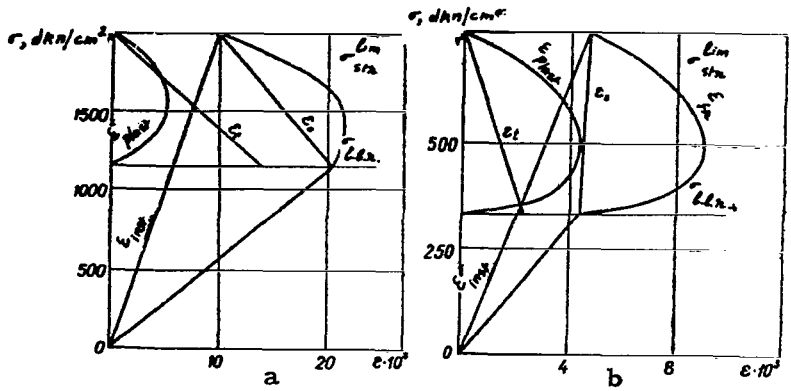

Fig. 20. Relation between stresses and deformations in the case of prolonged compression of DSP-B plastic along the ply layers along the fibers of lining (a) and pine wood along the fibers (b).

It is seen from these curves that instantaneous elastic deformation ε_{inst} grows linearly with an increase in the stresses. Elasto-viscous deformation ε_t developing with time decreases with an increase in stresses from a maximum value to zero at a stress corresponding to the limit of ultimate strength. The value of elastic deformation developing with time depends linearly on the magnitude of stresses.

Plastic deformation ε_{plast} caused by stresses equal to the limit of ultimate strength is equal to zero (brittle fracture). If the stresses are increased from a value equal to the limit of long-term resistance to a value equal to the ultimate strength, plastic deformation, at first, increases from zero to a definite value and if the stresses are increased still further, it falls from the maximum value to zero and the limiting elastic deformation changes according to the linear law. At stresses corresponding to the limit of ultimate strength, the values of instantaneous elastic and limiting elastic deformations at failure are equal. At stresses equal to the limit of long-term resistance, the limiting elastic deformation and deformation at failure are equal in magnitude.

CHAPTER IV

DEFORMABILITY AND RESISTIBILITY IN STEPPED AND CONTINUOUS LOADING AT CONSTANT RATES

MECHANISM OF STEPPED LOADING

Let us consider the case of stepped loading when the magnitude of loading at each step is equal and the interval of holding the loads at each step is also equal. Let us assume that the maximum value of stresses in stepped loading do not give rise to plastic deformations, i.e., $\sigma < \sigma_s$, where σ is equal to n steps of the stresses σ_1.

We shall consider, that the load at each step is acting continuously for a period of time which is equal to that for which it is held at that step. The instantaneous elastic deformation is equal in magnitude at each step of loading and in the case of stepped loading, it increases in proportion to the number of steps. The law of increase in elasto-viscous deformation in stepped loading is determined, on the one hand, by the law of increase in elasto-viscous deformation at continuously acting load within the limit of the given time interval equal to the time of holding of the load at the step, and on the other hand, by the influence of the action of the previous load. Influence of this load on the law of development of deformation at the given step of loading has been named by us as the heredity of the action of the previous load, in accordance with the heredity creep theory of Boltzman.

The development of elasto-viscous deformations at the given step of loading adds up with the continued development of elastic deformations due to the preyious steps of loading.

The law of development of elasto-viscous redeformations with time in stepped unloading is the same as the law of development of elasto-viscous deformations with time, but besides elasto-viscous redeformation in the course of the given step of unloading, an elasto-viscous deformation due to the remaining loads

also takes place. This deformation, when added up with redeformation, significantly changes its value and also the law of formation with time.

Thus, elasto-viscous redeformation developing with time during stepped unloading is also influenced by heredity of the loads during loading and unloading.

Heredity of the loading steps gives rise to development of elasto-viscous deformations in the course of a time, greater than the time of influence of heredity of unloading steps causing the development of redeformation. Hence, in the elastic region of work of material, the cycle of stepped loading and unloading is accompanied by the formation of a residual elastic deformation and the loop of elastic hysteresis is broken.

Heredity of the steps of unloading continues to appear after complete unloading, and on the expiry of a definite time interval, the residual elastic deformation under the influence of the continuing redeformation diappears as a result of the aftereffect and the loop of elastic hysteresis becomes closed. The aftereffect may be considered as a result of heredity, caused by the previous loading and unloading. The process of deformation in the elastic region of work of material in stepped loading and unloading,described above, is shown schematically in Fig. 21 in terms of the coordinates ε and t. Increase in deformations characterizes the influence of heredity in loading and unloading (broken line).

In order to confirm the process of deformation, considered above, experiments were conducted on DSP-plastics and wood. Experimental diagram of deformations in stepped loading and unloading of DSP-plastic is presented in Fig. 22. The diagram is plotted in the coordinates ε and t, and shows that the growth of the instantaneous elastic deformations follows Hooke's law, but the growth of elasto-viscous deformations essentially depends on the influence of heredity of the loads of the previous steps on the following steps. This is all the more clearly seen in Fig. 23, a, where the stresses are plotted along the ordinate and the growth of instantaneous and elasto-viscous deformations along the abscissa. At a constant value of the increase in stresses, the growth of deformations together with the increase in the serial number of the step takes place in accordance with a nonlinear law in the case of loading as well as unloading.

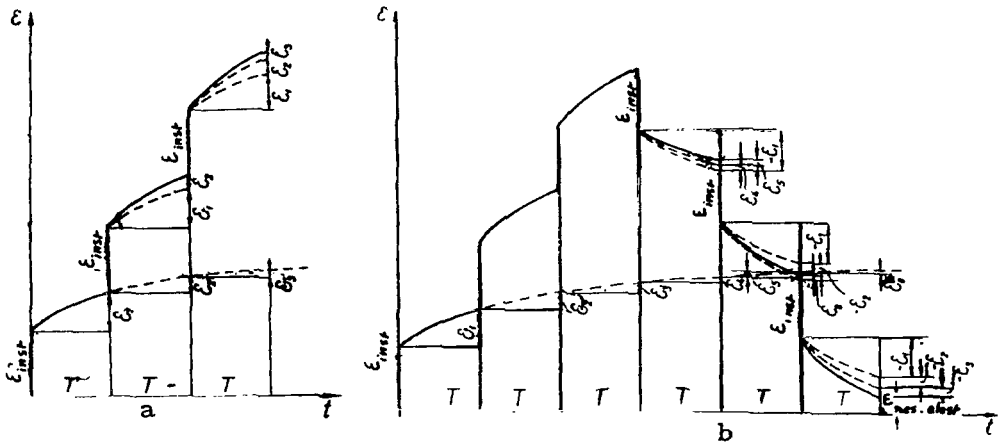

Fig. 21. Diagram of deformations in the case of stepped loading in the elastic region of work of material.
a - loading; b - unloading.

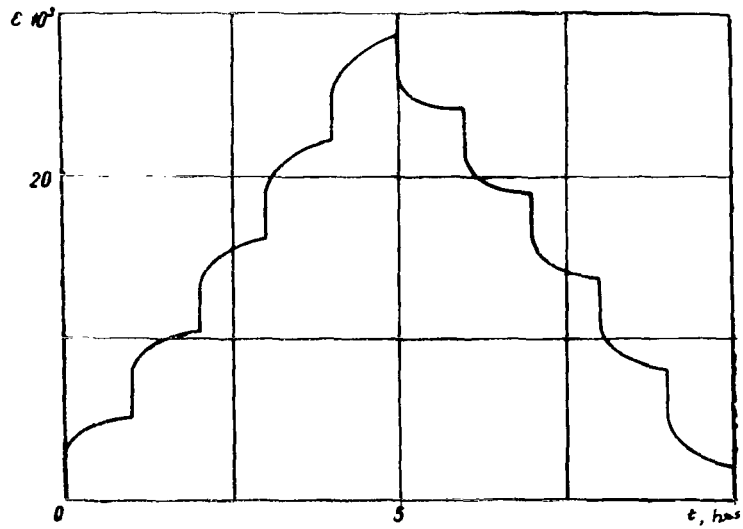

Fig. 22. Diagram of deformations in the case of stepped loading and unloading in the elastic region of work of material when DSP-B is compressed along the ply layers transverse to the fiber of the lining (σ_1 = 100 dkn/cm^2, T = 1 hour).

In order to clarify the laws of the formation and development of deformations in stepped loading in the elasto-plastic region of work of material, we conducted experiments on DSP-plastic and wood subjected to compression. In the Figs. 23,b and 23,c, typical diagrams of growth of deformation in stepped load-

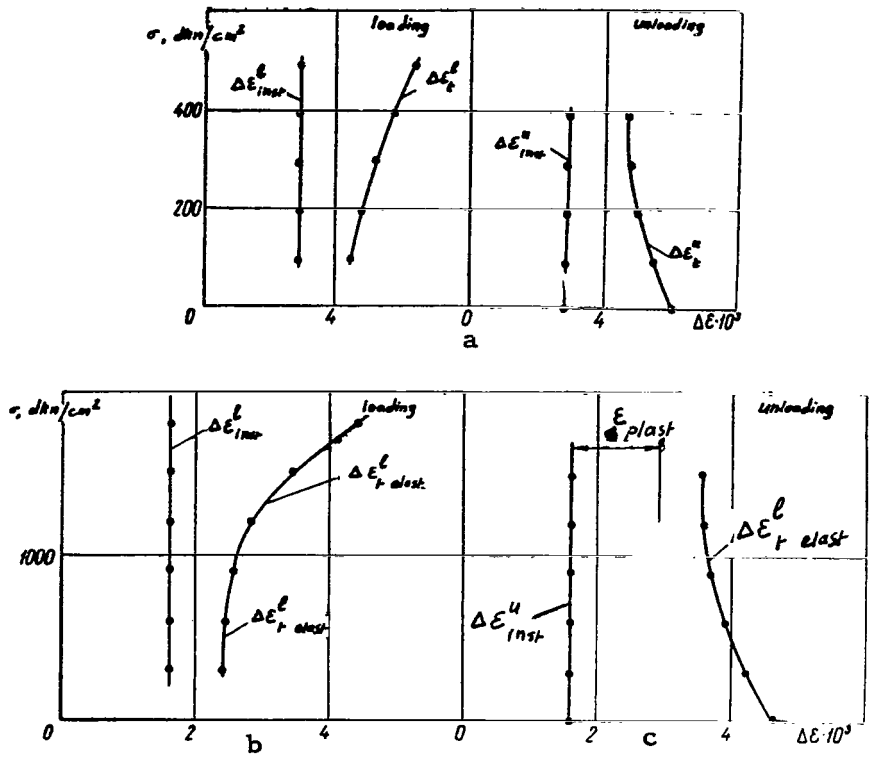

Fig. 23. Diagram of growth of deformations in the case of stepped loading and unloading when DSP-B plastic is compressed along the layers of the ply transverse to the fibers of the lining.

a - elastic region of work of material (σ_i = 100 dkn/cm², T = 1 hour);
b - elasto-plastic region of work of material (σ_i = 300 dkn/cm², T = 3 hours).

ing and unloading in the elasto-plastic region of work of material are shown in the coordinates σ and ε. As seen, instantaneous elastic deformations in stepped loading and unloading in the elasto-plastic as well as the elastic region of the work of material obey Hooke's law. Deformations developing with time in the elasto-plastic region of work of material, as well as in the elastic region grow according to a nonlinear law in the course of the entire time of action of the load at each step. But the growth of deformations in the elasto-plastic region differs from the growth of deformations in the elastic region. In the elasto-plastic region of work of material, the increase in deformations with time takes place considerably more intensely than in the elastic region, and as the serial number of the loading step goes up, the rate of increase in deformations also increases. The residual deformation after the loading-unloading cycle in this case con-

sists of elastic and plastic deformations. Elastic residual deformation gets totally redeformed in the course of a definite time interval after unloading, and the plastic residual deformation accumulated during the steps of loading and unloading does not change with time.

As a result of the experimental investigations, the mechanism of formation and development of deformations in the elasto-plastic region of work of material may be put in the following form. Instantaneous elastic deformations due to loading and unloading, which depend upon the value of the stresses acting at the steps, follow a linear law. The change in the elastic part of deformations, developing with time, in the elasto-plastic region of work of material follows the same rule as in the elastic region. The plastic part of deformations, developing with time, in the elasto-plastic region of work of material increases simultaneously with elasto-viscous deformations immediately following the instantaneous elastic deformations under the action of the total stresses, attained at the step of loading, at a rate corresponding to the stresses.

In the elasto-plastic region of work of material, the stresses acting at the step of loading are higher than the elastic limit from the very beginning of the development of deformations. Consequently, it may be considered, that in the elasto-plastic region of work of material, the plastic deformation is laid upon the elastic deformation in the course of the entire time of holding the load. This deformation develops with time according to a linear law, but the rate of its growth with an increase in stresses in the steps of loading increases according to a nonlinear law (see Chapter III).

While unloading after the preliminary loading to a stress $\sigma > \sigma_s$ by decreasing the stress, at first there takes place an instantaneous elastic redeformation equal to the instantaneous elastic deformation and then an elasto-viscous redeformation occurs at each step of unloading in the course of the time of holding the load. The law of formation and development of elasto-viscous redeformation with time in the elasto-plastic region of work of material and the nature of the influence of heredity of the preceding steps of loading and unloading is the same as in the case of loading and unloading in the elastic region.

On unloading to a stress level which demarcates the elastic and elasto-plastic regions of work of material, the plastic de-

formation attains a maximum value and does not change during the following unloading.

The values of elastic limits on loading and unloading differ from each other which is due to the influence of elasto-viscous deformations developing with time, heredity and accumulation of plastic deformations. On the whole, in the linear relation between the elastic limit and the corresponding limiting elastic deformation, all the abovementioned factors lead to the result that the elastic limit on unloading is lower than that on loading. In this case, there is an elasto-plastic hysteresis which, after complete unloading, is accompanied by the formation of residual deformations consisting of two parts: the plastic and the elastic.

The plastic part of residual deformations is equal to the sum of the plastic deformations in the steps of loading and unloading in the elasto-plastic region of work of material. The plastic part of residual deformation does not change with time in the process of the change in residual elastic deformation.

The elastic part of residual deformation after complete unloading decreases with time and becomes equal to zero after a definite time interval.

The abovementioned mechanism of formation and development of deformations in the case of stepped loading and unloading makes it possible to describe the process analytically in a comparatively simpler way by using the relationship valid for the continuous loading process.

STEPPED LOADING IN THE ELASTIC REGION OF WORK OF MATERIAL

Let us consider the deformation process in the case of stepped loading. Let us assume that the value of the step of stresses σ_1 and the time of holding T of the load are constant at every step.

Deformation at the first step of loading by stresses σ_1 and on holding the load for a time interval T is

$$\varepsilon^{1\to}_{total} = \frac{\sigma_1}{E_{prel}} \left[1 - Re^{-a\left(\frac{T}{t_1}\right)^c} \right]. \qquad (4.1)$$

As a result of the heredity of the first step the elasto-viscous deformation in the second step, developing in the course of the second time interval and caused by the load of the first step of loading, is added to the elasto-viscous deformation developing with time in the second step of loading. These deformations, when added together, give the total elasto-viscous deformations at the second step of loading, which also include the deformations caused by heredity of the action of first step of loading.

The total deformation at the end of time of holding the load at the second step of loading is

$$\varepsilon_{total}^{2\rightarrow} = \frac{\sigma_1}{E_{prol}}\left[2 - k\left(e^{-\alpha\left(\frac{2T}{t_1}\right)^c} + e^{-\alpha\left(\frac{T}{t_1}\right)^c}\right)\right]. \quad (4.2)$$

In further stepped loading, the magnitude of total deformations consists of instantaneous deformations, elasto-viscous deformations developing with time and elasto-viscous deformations caused by the heredity of each step of loading:

$$\varepsilon_{total}^{n\rightarrow} = \frac{\sigma_1}{E_{prol}}\left[n + k\left(1 - \sum_{a=0}^{a=n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right)\right]. \quad (4.3)$$

Let us consider the relation between the stresses and deformations caused by stepped unloading of the stresses σ_1 (same as in loading) and of equal time intervals T. In stepped unloading, it is essential to consider the instantaneous elastic redeformation obeying Hooke's law, elasto-viscous redeformation developing with time, elasto-viscous redeformation caused by the heredity of the previous steps of unloading, and elasto-viscous deformation caused by the heredity of the remaining steps of loading at the given unloading step.

Deformation, caused by the steps of loading, is determined from formula (4.3). The value of maximum stresses attained on unloading is

$$\sigma = n\sigma_1.$$

Let us unload the stresses σ_1 by one step and hold the remaining load at this step, giving rise to stresses (n-1) σ_1 at this step for a time interval T.

The value of redeformation on unloading by one step and the time T are determined as the difference between the redeformation on unloading by one step and the deformation caused by heredity of the remaining load by holding it for the same interval T.

Redeformation on unloading by one step is

$$\varepsilon^{1\leftarrow} = -\frac{\sigma_1}{E_{prol}}\left[1-ke^{-\alpha\left(\frac{T}{t_1}\right)^c}\right]. \qquad (4.4)$$

Deformation caused by the influence of heredity of the remaining load is

$$\varepsilon^{n\rightarrow 1\leftarrow}_{heredity} = \frac{\sigma_1 k}{E_{prol}}\left(\sum_{a=0}^{a=n+1} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \sum_{a=0}^{a=n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \right.$$

$$\left. - \sum_{a=0}^{a=1} e^{-\alpha\left(\frac{2aT}{t_1}\right)^c} + 1\right). \qquad (4.5)$$

Redeformation on unloading by one step is

$$\varepsilon^{1\leftarrow}_{total} = \frac{\sigma_1}{E_{prol}}\left[1-k\left(e^{-\alpha\left(\frac{T}{t_1}\right)^c} - e^{-\alpha\left[\frac{(n+1)T}{t}\right]^c} + e^{-\alpha\left(\frac{2T}{t_1}\right)^c}\right)\right] \qquad (4.6)$$

The total deformation after n steps of loading and one step of unloading is

$$\varepsilon^{n\rightarrow 1\leftarrow}_{total} = \frac{\sigma_1}{E_{prol}}\left[n-1-k\left(\sum_{a=0}^{a=n+1} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \sum_{a=0}^{a=2} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right)\right]. \qquad (4.7)$$

The equation for total deformation due to m steps of unloading after n steps of loading may be obtained in an analogous way.

Redeformation on unloading is determined from formula (4.3) in which m steps of unloading are put instead of n steps of loading:

$$\varepsilon^{m\leftarrow} = -\frac{\sigma_1}{E_{prol}}\left[m + k\left(1 - \sum_{a=0}^{a=m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right)\right]. \qquad (4.8)$$

Deformation caused by the influence of heredity on loading and unloading is

$$\varepsilon^{n\rightarrow m\leftarrow}_{heredity} = \frac{\sigma_1 k}{E_{prol}}\left(\sum_{a=0}^{a=n+m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \sum_{a=0}^{a=m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \right.$$

$$\left. - \sum_{a=0}^{a=m} e^{-\alpha\left(\frac{2aT}{t_1}\right)^c} + 1\right). \qquad (4.9)$$

Total deformation is

$$\varepsilon^{m\leftarrow}_{total} = -\frac{\sigma_1}{E_{prol}}\left[m - k\left(\sum_{a=0}^{a=m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \sum_{a=0}^{a=n+m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} + \right.\right.$$

$$\left.\left. + \sum_{a=0}^{a=n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} + \sum_{a=0}^{a=m} e^{-\alpha\left(\frac{2aT}{t_1}\right)^c} - 2\right)\right]. \qquad (4.10)$$

The remaining total deformation after loading by n steps and unloading by m steps is

$$\varepsilon^{n\rightarrow m\leftarrow}_{total} = \frac{\sigma_1}{E_{prol}}\left[n - m + k\left(\sum_{a=0}^{a=m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \sum_{a=0}^{a=n+m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} + \right.\right.$$

$$\left.\left. + \sum_{a=0}^{a=n} e^{-\alpha\left(\frac{2aT}{t_1}\right)^c} - 1\right)\right]. \qquad (4.11)$$

The value of the residual elastic deformation after loading and total unloading (m = n) may be determined from the formula (4.11). After transformations, we finally get

$$\varepsilon_{m\rightleftarrows n}^{res.\,elast} = \frac{\sigma_1 k}{E_{prol}} \left(\sum_{a=1}^{a=n} e^{-\alpha\left(\frac{2aT}{t_1}\right)^c} - \sum_{a=n+1}^{a=2n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} \right). \qquad (4.12)$$

Thus, after the cycle of stepped loading and unloading in the elastic region of work of material, the loop of elastic hysteresis opens and the gap is determined by the value of the residual elastic deformation. This discontinuity of the loop of elastic hysteresis is explained by the fact that during the test, elasto-viscous deformations accumulate under the influence of heredity on loading and unloading. The residual deformations after the completion of the cycle of loading and unloading continue to decrease with time until their values are reduced to zero, and the loop of elastic hysteresis is not closed. Closing of the loop of elastic hysteresis is the result of the continuing elastic redeformation with time after complete unloading.

The process of development of elasto-viscous redeformation after complete unloading has been named by us as the process of aftereffect. It shows how the deformation of a material without the load changes with time due to its previous action. Let us determine the time of aftereffect, essential for complete closure of the loop of elastic hysteresis. For this, we divide the time of aftereffect τ into intervals equal to the accepted intervals T for which the load is held during loading and unloading. Deformations due to elastic aftereffect consist of only elasto-viscous deformations developing with time. Instantaneous elastic deformations are not considered in this case, since they are directly related to the process of loading and unloading in the steps. In the first time interval T after complete unloading, the deformation due to aftereffect is equal to the difference in elasto-viscous deformations at the (m + 1)th and mth steps of loading.

The equation for redeformation in the course of the first time interval T after complete unloading may be obtained from the formula (4.3) when the number of steps of loading is equal to zero and the number of time intervals is m:

$$\varepsilon_{aft}^{t=T} = -\frac{\sigma_1 k}{E_{prol}} \left(\sum_{a=0}^{a=m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - 1 - m \right). \qquad (4.13)$$

Elasto-viscous deformation, after complete unloading under the influence of the aftereffect in the course of q intervals of time T, is obtained as the difference of redeformation developing in the course of the time interval n + q, taking the aftereffect in the course of the time interval q into consideration, and the redeformation on unloading:

$$\left.\begin{aligned} \varepsilon^{n\leftarrow}_{t=(n+q)T} &= \frac{\sigma_1 k}{E_{prol}}\left(n - \sum_{a=q+1}^{a=n+q} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right), \\ \varepsilon^{n\leftarrow}_{t=nT} &= \frac{\sigma_1 k}{E_{prol}}\left[n+1 - \sum_{a=0}^{a=n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right] \end{aligned}\right\} \qquad (4.14)$$

Deformation of the elastic aftereffect in the course of time $\tau = qT$ is

$$\varepsilon^{\tau}_{aft} = \frac{\sigma_1 k}{E_{prol}}\left(\sum_{a=1}^{a=n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \sum_{a=q+1}^{a=n+q} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right)$$

or

$$\varepsilon^{\tau}_{aft} = -\frac{\sigma_1 k}{E_{prol}}\left(e^{-\alpha\left(\frac{qT}{t_1}\right)^c} - 1\right)\left(\sum_{a=0}^{a=n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - 1\right) \qquad (4.15)$$

By equating the redeformation in the course of time interval τ after complete unloading to the value of the remaining elastic deformation, we get

$$\sum_{a=1}^{a=n} e^{-\alpha\left(\frac{2aT}{t_1}\right)^c} - \sum_{a=n+1}^{a=2n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} =$$

$$= \left(1 - e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right)\left(\sum_{a=0}^{a=n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - 1\right) \qquad (4.16)$$

Since $qT = \tau$, by solving the equation (4.16) with respect to τ, we get the time of aftereffect necessary for closing the loop of elastic hysteresis

$$\tau = qT = t_1 \sqrt[c]{\frac{1}{\alpha} \ln \frac{\sum_{a=0}^{a=h} e^{-\alpha\left(\frac{2aT}{t_1}\right)^c} - \sum_{a=0}^{a=2n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}}{\sum_{a=1}^{a=h} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}}} \qquad (4.17)$$

The process of aftereffect takes place in an irregular way at a rate decreasing with time. Actually, by considering the growth of the deformation of aftereffect in passing from the time interval $q > p > 1$ to $q > (p+1) > 1$, we find that the growth of the deformation of aftereffect in the course of one time interval T is

$$\Delta\varepsilon_{aft}^{T} = \frac{\sigma_1 k}{E_{prol}} \left(1 - e^{-\alpha\left(\frac{T}{t_1}\right)^c}\right) \left(\sum_{a=0}^{a=h} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - 1\right) e^{-\alpha\left(\frac{pT}{t_1}\right)^c} \qquad (4.18)$$

It is seen from equation (4.18) that all the terms except $e^{-\alpha\left(\frac{pT}{t_1}\right)^c}$ are constants and are independent of the time interval p of aftereffect under consideration. Hence, with the increase in the number of subsequent time intervals of aftereffect p, the magnitude of $e^{-\alpha\left(\frac{pT}{t_1}\right)^c}$ decreases, causing a reduction in the growth of deformation of aftereffect, i.e., the rate of increase of deformation of aftereffect decreases with time.

STEPPED LOADING IN THE ELASTO-PLASTIC REGION OF WORK OF MATERIAL

We shall select the value of the stresses σ_1 in such a way that the value of the total limiting elastic deformations and the corresponding stresses is reached at steps of loading with whole numbers. If it is impossible to achieve this in practice, then in the case of common nature of the change of deformations identical with that mentioned above, the difference will be only in the value of deformation at that step of loading, at which the transition takes place from elastic to elasto-plastic region of work of material. Since the value of plastic deformation at this step of loading is not large as compared to the following step, it

will not have considerable influence on the laws of formation and development of deformations in the elasto-plastic region of work of material in the case of stepped loading.

Let us assume, that the elastic limit σ_s is attained in n steps of stress σ_1, i.e.

$$\sigma_s = n\,\sigma_1 ,$$

and the value of total elastic deformations is equal to the limiting deformation:

$$\varepsilon_{total}^{n\rightarrow} = \varepsilon_s .$$

The value of the step of stresses in this case is

$$\sigma_1 = \frac{\sigma_{eff}}{n + (1-k_2)\left(1 - \sum_{a=0}^{a=n} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right)} . \qquad (4.19)$$

Let us assume that, in the case of stepped loading in the elasto-plastic region of work of material, elastic deformations develop according to the same law as in the elastic region. Then, at the (n+m)th step from the beginning of loading, the total elastice deformation is

$$\varepsilon_{total\ elast}^{n+m\rightarrow} = \frac{\sigma_1}{E_{prob}} \left[n+m+k\left(1 - \sum_{a=0}^{a=n+m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right)\right] . \qquad (4.20)$$

Development of plastic deformation proceeds simultaneously with the development of elastic deformations taking place after the elastic limit is attained.

Plastic deformation at the (n+m)th step of loading in the course of the entire time of holding the load at this step develops at a constant rate, corresponding to the attained level of stresses

$$\varepsilon_{Plast}^{n+m\rightarrow} = v_s^{n+m\rightarrow} T . \qquad (4.21)$$

The value of $v_s^{n+m\rightarrow}$ may be obtained from the formula (3.22), in which we put the quantity $\sigma_s = \sigma_1(n+m)$. By utilizing the value of the rate obtained in this way, we determine the magnitude of plastic deformation at the (n+m)th step of loading from the formula

$$\varepsilon_{plast}^{n+m\rightarrow} = \frac{kT[\sigma_1(n+m) - \sigma_{aft}]\sqrt[c]{\alpha}}{(1-k_2)t_1 E_{prol}} \left\{ \ln \frac{1-k_2}{1-\frac{\sigma_{aft}}{\sigma_1(n+m)}} \right\}^{1-\frac{1}{c}} \quad (4.22)$$

Summing up the values of plastic deformations at each step of loading, we obtain the value of total plastic deformation

$$\varepsilon_{total\ plast}^{n+m\rightarrow} = \frac{kT\sqrt[c]{\alpha}}{(1-k_2)t_1 E_{prol}} \sum_{a=1}^{a=m} [\sigma_1(n+a) - \sigma_{aft}] \times$$

$$\times \left[\ln \frac{1-k_2}{1-\frac{\sigma_{aft}}{\sigma_1(n+a)}} \right]^{1-\frac{1}{c}} \quad (4.23)$$

The total deformation at the (n+m)th step of loading is equal to the sum of elastic and plastic deformations:

$$\varepsilon_{total}^{n+m\rightarrow} = \frac{\sigma_1}{E_{prol}} \left[m+n+k\left(1-\sum_{a=0}^{a=n+m} e^{-\alpha\left(\frac{aT}{t_1}\right)^c}\right)\right] +$$

$$+ \frac{kT\sqrt[c]{\alpha}}{(1-k_2)t_1 E_{prol}} \sum_{a=1}^{a=m} [\sigma_1(n+a) - \sigma_{aft}] \left\{ \ln \frac{1-k_2}{1-\frac{\sigma_{aft}}{\sigma_1(n+a)}} \right\}^{1-\frac{1}{c}} \quad (4.24)$$

While unloading after loading up to the stresses $\sigma_1(n+m) > \sigma_s$, further elastic deformations occur under the influence of heredity of the remaining load and give rise to formation of plastic deformations. Besides, redeformation of elastic deformations

takes place on account of heredity on unloading. This process continues till the elastic limit and the corresponding deformation in unloading are reached.

By loading up to stresses $\sigma_1 (n+m) > \sigma_s$ and subsequent unloading in the elasto-plastic region of work of material by p steps, the value of the total elastic deformation may be determined by the formula (4.11), in which (m+n) and p are put instead of the number of steps of loading n and the number of steps of unloading m respectively.

While loading to stresses $\sigma_1 (n+m)$ and subsequent unloading by p steps, the formula for determining the elastic part of total deformation has the form:

$$\varepsilon^{\overset{n+m\rightarrow}{p\leftarrow}}_{total\ elast} = \frac{\sigma_1}{E_{prol}} \left[n+m-p+k \left(\sum_{a=0}^{a=p} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \right.\right.$$

$$\left.\left. - \sum_{a=0}^{a=n+m+p} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} + \sum_{a=0}^{a=p} e^{-\alpha\left(\frac{2aT}{t_1}\right)^c} - 1 \right) \right] \qquad (4.25)$$

Plastic deformation at the pth step of unloading is

$$\varepsilon^{p\leftarrow}_{plast} = \frac{kT\sqrt[c]{\alpha}\left[\sigma_1 (n+m-p) - \sigma_{aft}\right]}{(1-k_2)\, t_1 E_{prol}} \left[\ln \frac{1-k_2}{1-\frac{\sigma_{aft}}{\sigma_1 (m+n-p)}} \right]^{1-\frac{1}{c}} \qquad (4.26)$$

By summing up the values of plastic deformations at p steps of unloading, we obtain total plastic deformation on unloading:

$$\varepsilon^{p\leftarrow}_{total\ plast} = \frac{kT\sqrt[c]{\alpha} \sum_{a=1}^{a=p} \left[\sigma_1 (n+m-a)\, \sigma_{aft}\right]}{(1-k_2)\, t_1 E_{prol}} \left[\ln \frac{1-k_2}{1-\frac{\sigma_{aft}}{\sigma_1 (n+m-a)}} \right]^{1-\frac{1}{c}} \qquad (4.27)$$

The total amount of plastic deformation on loading by (n+m) steps and unloading by p steps is

$$\varepsilon^{p\leftarrow}_{total\ plast} = \frac{kT\sqrt[c]{\alpha}}{(1-k_2)t_1 E_{prol}} \times$$

$$\times \left[\sum_{a=1}^{a=m} [\sigma_1(n+a) - \sigma_{aft}] \left(\ln \frac{1-k_2}{1-\frac{\sigma_{aft}}{\sigma_1(n+a)}} \right)^{1-\frac{1}{c}} + \right.$$

$$\left. + \sum_{a=1}^{a=p} [\sigma_1(n+m-p) - \sigma_{aft}] \left(\ln \frac{1-k_2}{1-\frac{\sigma_{aft}}{\sigma_1(n+m-a)}} \right)^{1-\frac{1}{c}} \right], \qquad (4.28)$$

and total deformation is

$$\varepsilon^{p\leftarrow}_{total} = \frac{\sigma_1}{E_{prol}} \left[n+m-p+k \left(\sum_{a=0}^{a=p} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} - \right. \right.$$

$$\left. \left. - \sum_{a=0}^{a=n+m+p} e^{-\alpha\left(\frac{aT}{t_1}\right)^c} + \sum_{a=0}^{a=p} e^{-\alpha\left(\frac{2aT}{t_1}\right)^c} - 1 \right) \right] +$$

$$+ \frac{kT\sqrt[c]{\alpha}}{(1-k_2)t_1 E_{prol}} \left| \sum_{a=1}^{a=m} [\sigma_1(n+a) - \sigma_{aft}] \left(\ln \frac{1-k_2}{1-\frac{\sigma_{aft}}{\sigma_1(n+a)}} \right)^{1-\frac{1}{c}} + \right.$$

$$\left. + \sum_{a=1}^{a=p} [\sigma_1(n+m-p) - \sigma_{aft}] \left(\ln \frac{1-k_2}{1-\frac{\sigma_{aft}}{\sigma_1(n+m-a)}} \right)^{1-\frac{1}{c}} \right| \qquad (4.29)$$

On unloading the formation of plastic deformations continues till the value of total deformation becomes equal to the elastic limit on unloading, i.e.

$$\varepsilon^{p\leftarrow}_{total} = \varepsilon'_s . \qquad (4.30)$$

On further unloading by r steps in the elastic region, the amount of total deformation may be determined from the formula (4.25) by putting the number of steps (p + r) instead of the number of steps p. The value of total plastic part of deformation $\varepsilon^{p\ \overleftarrow{n+m}\rightarrow}_{total\ plast}$ is determined from the formula (4.28).

After complete unloading, i.e., at p + r = n + m, the remaining deformation consists of two parts: residual elastic and plastic deformations.

The residual plastic deformation is formed at m steps of loading and at p steps of unloading. Its value, obtained in the process of loading and unloading, does not change with the lapse of time after complete unloading.

Residual elastic deformation takes place at all steps of loading and unloading as a result of the influence of heredity. Its value may be obtained from the formula (4.12), in which the value of steps is put as n + m.

The residual elastic deformation after the completion of the cycle of loading and unloading changes with time under the influence of aftereffect. The process of aftereffect takes place according to the same laws, as in the cycle of loading and unloading in the elastic region of work of material. The time interval of the aftereffect may be determined from formula (4,17), in which the number of steps n + m should be put in place of the cycle of steps n.

In the course of the time interval of the aftereffect, the residual elastic deformation gets totally redeformed and the remaining part of the deformation represents plastic deformation, which does not change its value with time, if the state of the material and the medium remains constant.

TESTS WITH A CONSTANT LOADING RATE IN ELASTIC REGION OF WORK OF MATERIAL

The study of the resistibility and deformability of elasto-visco-plastic materials by testing at constant loading rate makes it possible to establish a relation between the mechanical properties obtained in this experiment and the characteristics of prolonged strength and deformability. Besides, a number of other

problems of considerable importance in evaluating a material and in selecting the methods of its test can be solved. These, in the first place, include the problems of studying the influence of the test rate and the selection of the optimum rates, the study of the change and the relation of the characteristics of strength with proportional and elastic limits, investigation of the modulus of elasticity determined from the deformation diagram and the establishment of its relation with prolonged characteristics of deformability, the study of hysteresis, Baushinger effect etc.

The test with a constant rate of loading may be considered to be stepped loading with the value of step loading being $d\sigma$ and the period of loading being dt at every step.

Let us consider that the stresses change according to a linear law $\sigma = v_\sigma t$, where t is the test time and v_σ is the loading rate. In accordance with what has been stated above, the increase in deformation due to an increase in stresses $d\sigma$ is

$$d\varepsilon = \frac{d\sigma}{E_{prol}}\left[1 - ke^{-\alpha\left(\frac{t}{t_1}\right)^c}\right], \qquad (4.31)$$

where t is the time of development of deformation during loading.

Integrating the equation (4.31) within the limits 0 to σ and putting $d\sigma = v_\sigma dt$, we get the total deformation at the given loading rate v_σ corresponding to the final value of the stresses σ while loading as a function of time t. The value of total deformation on loading is

$$\varepsilon^{\sigma\rightarrow}_{total} = \frac{kv_\sigma}{E_{prol}}\left[\frac{t}{k} - \int_0^t e^{-\alpha\left(\frac{t}{t_1}\right)^c} dt\right] \qquad (4.32)$$

or

$$\varepsilon^{\sigma\rightarrow}_{total} = \frac{v_\sigma t}{E_{prol}} + \frac{kv_\sigma}{E_{prol}} e^{-\alpha\left(\frac{t}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} t^{1-nc} \qquad (4.33)$$

Replacing V_σ by σ/t which are equal, the equation (4.33) may be written as

$$\varepsilon^{\sigma\to}_{total} = \frac{\sigma}{E_{prol}} + \frac{k V_\sigma}{E_{prol}} e^{-\alpha\left(\frac{\sigma}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{\sigma}{V_\sigma}\right)^{1-nc} \qquad (4.34)$$

The equation (4.34) at values of loading rate $V_\sigma \to 0$ and $V_\sigma \to \infty$ gives equations of straight lines passing through the origin of the coordinates in the deformation diagram. This agrees with the absence or total elimination of the influence of viscosity.

Utilizing the results of the change in deformations in the case of stepped unloading, we obtain the value of change in deformation due to a change in the value of stresses at a constant rate.

Let us denote the time necessary to attain the maximum stresses σ on loading by t_ℓ and the time of the subsequent unloading by t_u. Then

$$d\varepsilon = \frac{d\sigma}{E_{prol}} \left(1 - k e^{-\alpha\left(\frac{t_u}{t_1}\right)^c}\right) - \frac{d\sigma k}{E_{prol}} \left(e^{-\alpha\left(\frac{t_\ell - t_u}{t_1}\right)^c} - e^{-\alpha\left(\frac{2t_u}{t_1}\right)^c}\right). \qquad (4.35)$$

Let us integrate within the limits 0 to σ_u, i.e., from 0 to t_u, where the time t_u is variable and the time of the preliminary loading in the unloading process t_ℓ is constant:

$$\varepsilon^{t_u}_{total} = \frac{V_\sigma}{E_{prol}} \int_0^{t_u} \left(1 - k e^{-\alpha\left(\frac{t_u}{t_1}\right)^c}\right) dt -$$

$$- \frac{V_\sigma k}{E_{prol}} \int_0^{t_u} \left(e^{-\alpha\left(\frac{t_\ell - t_u}{t_1}\right)^c} - e^{-\alpha\left(\frac{2t_u}{t_1}\right)^c}\right) dt. \qquad (4.36)$$

Finally we get

$$\varepsilon^{t_u}_{total} = \frac{V_\sigma t_u}{E_{prol}} + \frac{kV_\sigma}{E_{prol}} e^{-\alpha\left(\frac{t_u}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{1(1-c)\ldots.[1-(n-1)c]}{\left(\frac{\alpha c}{t_1^c}\right)^n} -$$

$$- \frac{V_\sigma k}{E_{prol}} e^{-\alpha\left(\frac{t_\ell + t_u}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{1(1-c)\ldots.[1-(n-1)c]}{\left(\frac{\alpha c}{t_1^c}\right)^n} (t_\ell + t_u)^{1-nc} -$$

$$- \frac{V_\sigma k}{E_{prol}} e^{-\alpha\left(\frac{2t_u}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{1(1-c)\ldots.[1-(n-1)c]}{\left(\frac{\alpha 2^c c}{t_1^c}\right)^n} t_u^{1-nc} . \qquad (4.37)$$

Putting in the formula (4.37) the values of t_ℓ and t_u as expressed in terms of stresses

$$t_\ell = \frac{\sigma}{V_\sigma} \quad \text{and} \quad t_u = \frac{\sigma_u}{V_\sigma} ,$$

we obtain the expression for the determination of deformation during unloading as a function of stresses

$$\varepsilon^{\sigma_u}_{total} = \frac{\sigma_u}{E_{prol}} \left\{ 1 + \frac{kV_\sigma}{\sigma_u} \left[e^{-\alpha\left(\frac{\sigma_u}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{\sigma_u}{V_\sigma}\right)^{1-nc} - \right.\right.$$

$$- e^{-\alpha\left(\frac{\sigma+\sigma_u}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{\sigma+\sigma_u}{V_\sigma}\right)^{1-nc} -$$

$$\left.\left. - e^{-\alpha\left(\frac{2\sigma_u}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha 2^c c}{t_1^c}\right)^n} \left(\frac{\sigma_u}{V_\sigma}\right)^{1-nc} \right] \right\}, \qquad (4.38)$$

where σ_u is the value of the stresses removed on unloading in the course of time after loading up to stresses σ .

The remaining total deformation after the removal of the stress σ_u during unloading as a function of unloading time t_u is equal to the difference of the total deformation during loading and the deformation during unloading:

$$\varepsilon^{t_\ell}_{t_u}_{total} = \frac{V_\sigma t_\ell}{E_{prol}} + \frac{kV_\sigma}{E_{prol}} e^{-\alpha\left(\frac{t_\ell}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} t_\ell^{1-nc} -$$

$$- \frac{V_\sigma t_u}{E_{prol}} - \frac{kV_\sigma}{E_{prol}} e^{-\alpha\left(\frac{t_u}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} t_u^{1-nc} +$$

$$+ \frac{kV_\sigma}{E_{prol}} e^{-\alpha\left(\frac{t_\ell + t_u}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} (t_\ell + t_u)^{1-nc} +$$

$$+ \frac{kV_\sigma}{E_{prol}} e^{-\alpha\left(\frac{2t_u}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha 2^c c}{t_1^c}\right)^n} t_u^{1-nc} \qquad (4.39)$$

or as a function of stresses

$$\varepsilon^{\sigma\rightarrow}_{\sigma_u\leftarrow}_{total} = \frac{\sigma}{E_{prol}} - \frac{\sigma_u}{E_{prol}} + \frac{kV_\sigma}{E_{prol}} \left[e^{-\alpha\left(\frac{\sigma}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{\sigma}{V_\sigma}\right)^{1-nc} - \right.$$

$$- e^{-\alpha\left(\frac{\sigma_u}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{\sigma_u}{V_\sigma}\right)^{1-nc} +$$

$$+ e^{-\alpha\left(\frac{\sigma+\sigma_u}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{\sigma+\sigma_u}{V_\sigma}\right)^{1-nc} +$$

$$\left. + e^{-\alpha\left(\frac{2\sigma_u}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha 2^c c}{t_1^c}\right)^n} \left(\frac{\sigma_u}{V_\sigma}\right)^{1-nc} \right]. \qquad (4.40)$$

At complete unloading, the remaining deformation is

$$\varepsilon^{\sigma\rightarrow}_{\sigma\leftarrow}{}_{total} = \frac{kV_\sigma}{E_{prol}} e^{-\alpha\left(\frac{2\sigma}{V_\sigma t_1}\right)^c} \left[\sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{2\sigma}{V_\sigma}\right)^{1-nc} + \right.$$

$$\left. + \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha 2^c c}{t_1^c}\right)^n} \left(\frac{\sigma}{V_\sigma}\right)^{1-nc} \right] \qquad (4.41)$$

or

$$\varepsilon^{\sigma\rightarrow}_{\sigma\leftarrow}{}_{total} = \frac{kV_\sigma}{E_{prol}} e^{-\alpha\left(\frac{2t}{t_1}\right)^c} \left[\sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} (2t)^{1-nc} + \right.$$

$$\left. + \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha 2^c c}{t_1^c}\right)^n} t^{1-nc} \right]. \qquad (4.42)$$

In the formulas given above, σ is the value of maximum stresses on loading.

A characteristic form of the diagram of deformations in the elastic region of work of material (loading and unloading curves) for the tests at a constant loading rate is shown in Fig. 24. The open nature of the loop of elastic hysteresis immediately after complete unloading is explained by the fact that the aftereffect resulting from the development of the elasto-viscous deformation occurring with time after unloading was not taken into consideration.

The elastic aftereffect due to the heredity influence during unloading leads to the result that the residual elastic deformation decreasing with time after complete unloading becomes equal to zero and the loop of elastic hysteresis closes. Closing of the loop of elastic hysteresis takes place during the course of the time interval τ, the value of which depends on the time of influence of heredity during loading and unloading.

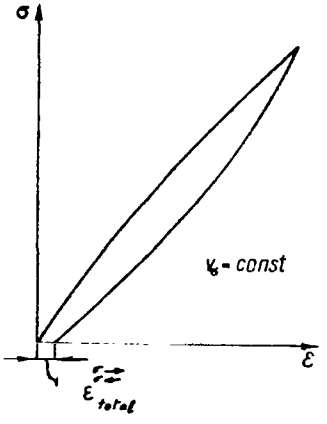

Fig. 24. Diagram of deformations by testing at a constant rate of loading in the elastic region of work of material.

The process of redeformation under the heredity influence or the process of aftereffect, takes place with time unevenly at a variable decreasing rate of redeformation in the course of time τ on the expiry of which the loop of elastic hysteresis closes.

Let us find out the value of deformation of aftereffect and establish the law of its change with time, using the rules of aftereffect in the case of stepped loading and unloading.

Using the equation (4.15), we obtain the growth of the deformation of aftereffect:

$$d\varepsilon_{aft} = \frac{V_\sigma R\, dt}{E_{prol}} \left(e^{-\alpha\left(\frac{\tau}{t_1}\right)^c} - e^{-\alpha\left(\frac{t_u+\tau}{t_1}\right)^c} \right), \qquad (4.43)$$

where τ is the time of aftereffect.

Redeformation after complete unloading as a function of τ or the deformation of aftereffect can be obtained as

$$\varepsilon^{\tau}_{aft} = \frac{R V_\sigma}{E_{prol}} \int_0^{\tau} \left(e^{-\alpha\left(\frac{\tau}{t_1}\right)^c} - e^{-\alpha\left(\frac{t_u+\tau}{t_1}\right)^c} \right) dt. \qquad (4.44)$$

Integrating, we get

$$\varepsilon^{\tau}_{aft} = -\frac{k V_\sigma}{E_{prol}} \left(e^{-\alpha\left(\frac{\tau}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \tau^{1-nc} + \right.$$

$$\left. + e^{-\alpha\left(\frac{t_u+\tau}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} (t_u+\tau)^{1-nc} \right). \qquad (4.45)$$

Equating the deformation, caused by aftereffect, to the residual elastic deformation, we get

$$e^{-\alpha\left(\frac{t_u+\tau}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} (t_u+\tau)^{1-nc} -$$

$$- e^{-\alpha\left(\frac{\tau}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \tau^{1-nc} =$$

$$= e^{-\alpha\left(\frac{2t_u}{t_1}\right)^c} \left[\sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} (2t_u)^{1-nc} + \right.$$

$$\left. + \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha 2^c c}{t_1^c}\right)} t_u^{1-nc} \right]. \qquad (4.46)$$

By solving the equation (4.46), the required time of total redeformation can be obtained.

The limiting elastic deformation at the given loading rate V_σ may be obtained from the formula (4.34), in which σ is equal to σ_s :

$$\varepsilon_s = \frac{\sigma_s}{E_{prol}} + \frac{kV_\sigma}{E_{prol}} e^{-\alpha\left(\frac{\sigma_s}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{\sigma_s}{V_\sigma}\right)^{1-nc}. \tag{4.47}$$

Considering the fact that at one and the same value of elastic limit, the corresponding deformation in the case of a test with a constant rate of loading is equal to deformation at a constant sustained load, after transformations we get:

$$\frac{\sigma_{l.l.} - \sigma_s}{1-k_2} - V_\sigma e^{-\alpha\left(\frac{\sigma_s}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{\sigma_s}{V_\sigma}\right)^{1-nc} = 0. \tag{4.48}$$

The equation (4.48) gives the relation between the magnitude of elastic limit and the rate of loading during the test.

A graphical solution of the equation (4.48) for the case of pine wood subjected to compression along fibers is given as an example, in Fig. 25, and Fig. 26 shows the relation between elastic limit and the loading rate (lower curve).

TESTS WITH A CONSTANT LOADING RATE IN ELASTO-PLASTIC REGION OF WORK OF MATERIAL

Let us consider the deformation process in the elasto-plastic region of work of material while testing at a constant loading rate. Total deformation in this case consists of elastic and plastic deformations. The development of plastic deformations at stresses $\sigma > \sigma_s$ takes place simultaneously with development of elastic deformations.

The growth of the plastic deformations, according to the formula (4.23), is

$$d\varepsilon_{plast} = \frac{k\alpha^{\frac{1}{c}} (\sigma - \sigma_{l.l.})\, d\sigma}{(1-k_2) t_1 V_\sigma E_{prol}} \left[\ln \frac{\sigma(1-k_2)}{\sigma - \sigma_{l.l.}}\right]^{\frac{c-1}{c}}, \tag{4.49}$$

where $d\sigma = V_\sigma\, dt$.

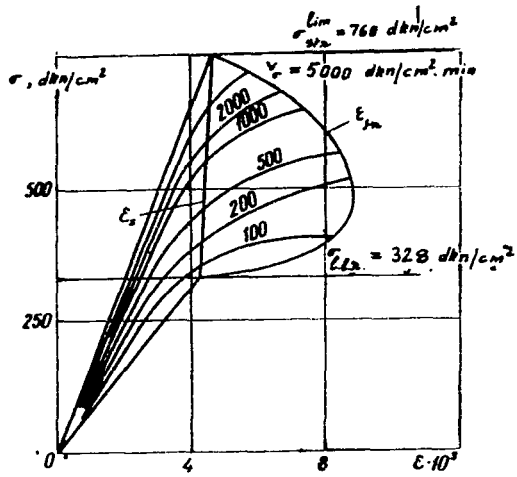

Fig. 25. Diagrams of deformations due to prolonged action of a constant load and tested at different loading rates.

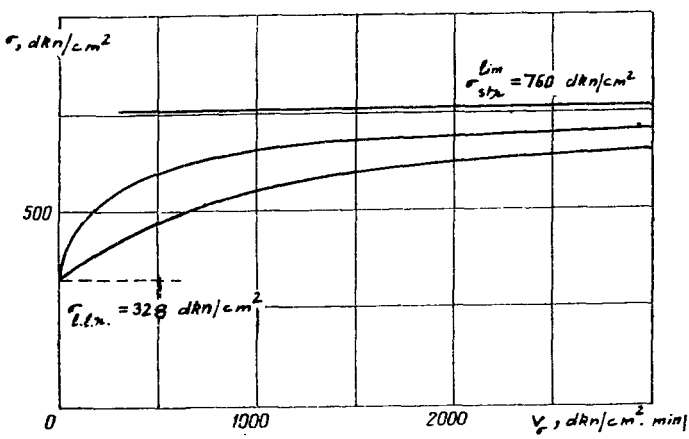

Fig. 26. Dependence of the ultimate strength (upper curve) and elastic limit (lower curve) on the rate of increase in stresses.

Integrating the equation (4.49) within the limits of the elastic limit σ_s and the maximum stresses on loading $\sigma \leq \sigma_{str}$ (σ_{str} is the ultimate strength of material at the given loading rate), we get the value of plastic deformation

$$\varepsilon_{plast} = \frac{k\alpha^{\frac{1}{c}} c}{(1-k_2)\, t_1 V_\sigma E_{prol}} \int_{\sigma_s}^{\sigma} (\sigma - \sigma_{l.l.n.}) \left[\ln \frac{\sigma(1-k_2)}{\sigma - \sigma_{l.l.n.}} \right]^{\frac{c-1}{c}} d\sigma. \qquad (4.50)$$

Since integration of the equation (4.50) is comparatively easy only for partial values (for example, when c = 1), it is expedient to substitute

$$\left[\ln\frac{\sigma(1-k_2)}{\sigma_s-\sigma_{l.l.n.}}\right]^{1-\frac{1}{c}}$$

by its approximate value which differs very little from the exact value within practically feasible limits. It is known from experiments that the value of the coefficient k_2 changes from $k_2 = 0.1$ to $k_2 = 0.9$, and stresses σ_s at the given rate of loading change from $\sigma_s = 1.1\,\sigma_{l.l.n.}$ to $\sigma_s = 0.90\,\sigma_{str}^{lim}$. By the method of selection of approximate values, it was established that it is possible in practice to substitute

$$\ln\frac{\sigma_s(1-k_2)}{\sigma_s-\sigma_{l.l.n.}} \approx \frac{\sigma_{l.l.n.}-k_2\sigma_s}{\sigma_s-\sigma_{l.l.n.}}.$$

This approximate value in the abovementioned ranges of k_2 and σ_s gives an error not higher than 10 to 12% in the worst combinations of k_2 and σ_s, and at usual values of k_2 and σ_s, this deviation is 3 to 5%. Using the approximate value, we obtain a formula for determining the plastic deformation in the case of tests with constant loading rates.

$$\varepsilon_{plast} = \frac{Rc\alpha^{\frac{1}{c}}}{(1-k_2)t_1E_{prol}V_\sigma}\int_{\sigma_s}^{\sigma}(\sigma_s-\sigma_{l.l.n.})\left[\frac{\sigma_{l.l.n.}-k_2\sigma_s}{\sigma_s-\sigma_{l.l.n.}}\right]^{1-\frac{1}{c}}d\sigma_s,$$

or

$$\varepsilon_{plast} = \frac{Rc\alpha^{\frac{1}{c}}}{(1-k_2)t_1E_{prol}V_\sigma}\int_{\sigma_s}^{\sigma}\frac{(\sigma_s-\sigma_{l.l.n.})^{\frac{1}{c}}}{(\sigma_{l.l.n.}-k_2\sigma_s)^{\frac{1}{c}-1}}d\sigma_s. \qquad (4.51)$$

The value of total deformation in tests with a constant loading rate is equal to the sum of elastic and plastic deformations:

$$\varepsilon^{\sigma\rightarrow}_{total} = \frac{\sigma}{E_{prol}} + \frac{kV_\sigma}{E_{prol}} e^{-\alpha\left(\frac{\sigma}{V_\sigma t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} \left(\frac{\sigma}{V_\sigma}\right)^{1-nc} +$$

$$+ \frac{kc\alpha^{\frac{1}{c}}}{(1-k_2)t_1 E_{prol} V_\sigma} \int_{\sigma_s}^{\sigma} \frac{(\sigma-\sigma_{e.e.n})^{\frac{1}{c}}}{(\sigma_{l.l.n.}-k_2\sigma)^{\frac{1}{c}-1}} d\sigma. \qquad (4.52)$$

At certain values of c, the equation (4.52) becomes appreciably simple. Thus, at c = 1,

$$\varepsilon^{\sigma\rightarrow}_{total} = \frac{\sigma}{E_{prol}} - \frac{kV_\sigma t_1}{E_{prol}} \left(1 - e^{-\alpha\left(\frac{\sigma}{V_\sigma t_1}\right)}\right) +$$

$$+ \frac{k\alpha(\sigma-\sigma_s)(\sigma+\sigma_s-2\sigma_{e.l.n.})}{2(1-k_2)E_{prol}V_\sigma}.$$

The above equations of the deformation diagrams are ascending curves without a falling part, which is confirmed by experiments on elasto-plastic materials with a constant loading rate.

Let us draw a conclusion from the equation of the deformation diagram for unloading after preliminary loading to stresses $\sigma_l > \sigma_s$.

On unloading, the elastic part of deformation due to decrease in stresses changes according to the same law as in the elastic region of work of material.

Using the formula (4.26), it is possible to obtain the change in the value of plastic deformation on unloading. If σ_n is the maximum value of stresses removed during unloading, the stresses causing the development of plastic deformation on unloading are $\sigma = \sigma_l - \sigma_u$ and in the process of test, they change within the limits $\sigma = \sigma_l$ and $\sigma \geqslant \sigma_s$.

Using the formula (4.26), we get the magnitude of plastic deformation on unloading:

$$\varepsilon_{Plast}^{\sigma_u \leftarrow} = \frac{RC\sqrt[c]{\alpha}}{(1-k_2)\,t_1 E_{prol}} \int_{\sigma_\ell}^{\sigma} (\sigma - \sigma_{\ell.\ell.n.}) \left[\ln \frac{1-k_2}{1-\frac{\sigma_{\ell.\ell.n.}}{\sigma}} \right]^{1-\frac{1}{c}} d\sigma \tag{4.53}$$

or, approximately

$$\varepsilon_{Plast}^{\sigma_u \leftarrow} = \frac{R\alpha^{\frac{1}{c}} C}{(1-k_2)\,t_1 E_{prol}} \int_{\sigma_\ell}^{\sigma} \frac{(\sigma - \sigma_{\ell.\ell.n.})^{\frac{1}{c}}}{(\sigma_{\ell.\ell.n.} - k_2\sigma)^{\frac{1}{c}-1}} d\sigma. \tag{4.54}$$

The total value of plastic deformation on loading to stresses $\sigma_\ell > \sigma_S$ and subsequent unloading up to stresses $\sigma_\ell > \sigma_u > \sigma_S$ is

$$\varepsilon_{Plast.\,total}^{\substack{\sigma_\ell \rightarrow \\ \sigma_u \leftarrow}} = \frac{R\alpha^{\frac{1}{c}} C}{(1-k_2)\,t_1 E_{prol}} \left[\int_{\sigma_S}^{\sigma_\ell} \frac{(\sigma - \sigma_{\ell.\ell.n.})^{\frac{1}{c}}}{(\sigma_{\ell.\ell.n.} - k_2\sigma)^{1-\frac{1}{c}}} d\sigma + \right.$$

$$\left. + \int_{\sigma_\ell}^{\sigma_u} \frac{(\sigma - \sigma_{\ell.\ell.n.})^{\frac{1}{c}}}{(\sigma_{\ell.\ell.n.} - k_2\sigma)^{1-\frac{1}{c}}} d\sigma \right]. \tag{4.55}$$

Total deformation after loading to the maximum stresses σ_ℓ and subsequent unloading to stresses σ_u is obtained as the sum of elastic and plastic deformations.

Formation of plastic deformations on unloading continues till the total deformation during loading attains a limiting value differing from its value on loading.

If the limiting elastic deformations during loading and unloading differ, the corresponding stresses will also differ in

their values. The condition of transition to elastic deformation on loading may be written as $\varepsilon^{\sigma_\ell \rightarrow}_{total} = \varepsilon'_s$ at $\varepsilon'_s \neq \varepsilon_s$ where ε_s is the limiting elastic deformation on loading.

The value of total plastic deformation in the cycle of loading and unloading may be obtained from the formula (4.55) by putting in it the values $\sigma'_u = \sigma'_s$, where σ'_s is the value of stresses corresponding to the limiting elastic deformation ε'_s. The values of σ'_s and ε'_s may be found by simultaneously solving the equation (3.19) and the equation of the change in total deformation on unloading.

The relation between the value of ultimate strength and the loading rate may be established by considering the equation of the deformation curve.

Assuming that at stresses $\sigma_\ell = \sigma_{str}$, the deformation $\varepsilon^{\sigma \rightarrow}_{total}$ is equal to the deformation at fracture ε_{fr} (3.30), we get

$$\frac{\sigma_{str}}{E_{prol}}\left|1-\frac{k}{1-k_2}\left(1-\frac{\sigma_{l.l.r.}}{\sigma_{str}}\right)\times\right.$$

$$\times\left\{1+\left(\frac{t_2\left\{\frac{1}{\alpha_1}\ln\frac{1-k_2}{\frac{\sigma_{str}}{\sigma_{l.l.r}}-1}\right\}^{\frac{1}{c_1}}}{t_1\left\{\frac{1}{\alpha}\ln\frac{1-k_2}{1-\frac{\sigma_{l.l.r}}{\sigma_{str}}}\right\}^{\frac{1}{c}}}-1\right)\ln\frac{1-\frac{\sigma_{l.l.r.}}{\sigma_{str}}}{1-k_2}\right\}\Bigg|=$$

$$=\frac{\sigma_{str}}{E_{prol}}\left[1+\frac{kv_\sigma}{\sigma_{str}}e^{-\alpha\left(\frac{\sigma_{str}}{v_\sigma t_1}\right)^c}\sum_{n=1}^{n=\infty}\frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n}\left(\frac{\sigma_{str}}{v_\sigma}\right)^{1-nc}\right]+$$

$$+\frac{kc\sqrt[c]{\alpha}}{(1-k_2)t_1E_{prol}v_\sigma}\int_{\sigma_s}^{\sigma_{str}}\frac{(\sigma-\sigma_{l.l.r.})^{\frac{1}{c}}}{(\sigma_{l.l.r.}-k_2\sigma)^{\frac{1}{c}-1}}d\sigma. \qquad (4.56)$$

Equation (4.56) gives the relation between the magnitude of ultimate strength and the loading rate.

As an example, a graphical solution of the equation giving the relation between the ultimate strength and the loading rate for pine wood subjected to compression along the fibers is presented in Fig.25; Fig. 26 shows the relation between the ultimate strength and the loading rate (upper curve).

CHAPTER V

EXPERIMENTAL INVESTIGATION OF MECHANICAL PROPERTIES OF LAMINATED PLASTICS AND WOOD BY TESTING AT CONSTANT RATE OF LOADING

METHOD OF WORKING OUT THE DEFORMATION DIAGRAM

Investigations of mechanical properties of plastic glasses and DSP-wood laminated plastic were conducted by short duration test at a given loading rate. In this case, the deformation diagram is a monotonically rising curve with three characteristic parts (Fig. 27).

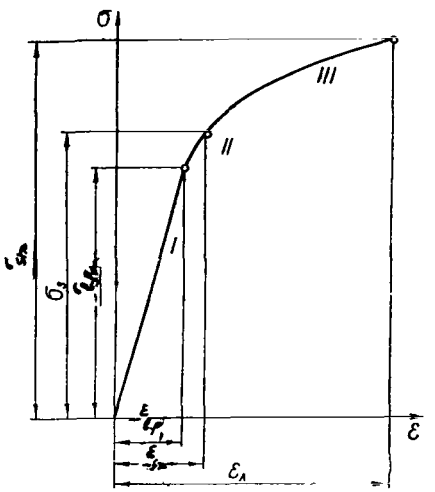

Fig. 27. Diagram of deformations by testing at a given loading rate.

Since, in part I, the tangent of the angle of inclination of the tangent to the deformation curve does not change appreciably, it is possible to replace the curve by an equivalent straight line, considering that in this part the material follows the law of proportionality.

The conditional limit of proportionality is represented by the boundary of part I in terms of stresses. The tangent of the

angle of inclination of the rectilinear part of the deformation diagram with the deformation axis determines the value of modulus of elasticity at the given loading rate.

Part II of the deformation diagram looks like a bulging curve. This part begins from the limit of proportionality and ends at the stress equal to the elastic limit. In part II of the deformation diagram, the formation of instantaneous elastic and elasto-viscous deformations takes place and the development of the elasto-viscous deformation is more intense than in Part I.

The value of the conditional limit of proportionality may be higher or lower than the value of elastic limit depending upon the allowance adopted while determining it. When the value of the conditional limit of proportionality is higher than the elastic limit, part II of the deformation diagram does not actually exist.

Part III of the deformation diagram is characterized by the simultaneous development of instantaneous elastic, elasto-viscous and plastic deformations.

In the case of compression of DSP-plastic, deformation diagrams were obtained from the points plotted by measuring the deformations due to stepped loading, imitating continuous loading at the given rate. This was achieved by holding the load at each step for a definite time interval corresponding to the given constant loading rate. Based on the possibilities of the machine of IM AN UkSSR[a/], 240 dkn/min was accepted.

Instantaneous elastic deformation at every step of loading at the beginning of the respective holding interval and at the end of this interval was determined by an indicator.

Instantaneous deformation diagram was plotted by measuring the deformation at the beginning of the interval during which the load was held, and the total deformation diagram was obtained from the measurements taken at the end of this interval. The values of the conditional limits of proportionality, elastic limit, ultimate strength, relative limiting elastic deformation, relative total deformation at the time of fracture, instantaneous modulus

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a/ Translator's Note: IM AN UkSSR - Institut Mekhaniki Akademii Nauk, UkSSR (Institut of Mechanics, Academy of Sciences of the Ukranian SSR).

of elasticity and modulus of elasticity at the linear part of the diagram were determined by processing these diagrams.

Total deformation diagrams of plastic glasses subjected to compression were obtained by a 30 t testing hydraulic machine of the type "Emery Baldwin" by continuous loading at such a constant rate of increase of load that the rate of growth of stresses is equal to or higher than the optimum value. Deformations were also measured by indicators on a base equal to the height of the specimen. While measuring deformations of the specimen, the total deformations, as in the case of testing DSP-plastic, also included the deformation of transverse crumpling, as a result of which deformations of the specimens were higher to some extent.

Conditional limit of proportionality was defined as the stresses, at which the modulus of elasticity on the curved part of the deformation diagram decreases by 50% as compared to the modulus of elasticity in the conditional linear part. For any arbitrarily selected stresses within the range of the rectilinear part of the deformation diagram, the deformation corresponding to it was taken by increasing it by 50%. Then, through the origin of the coordinates of the deformation diagram a straight line was drawn, which passed through the point corresponding to the increased value of deformation. A tangent was then drawn to the deformation curve, parallel to this straight line. The point of contact defines the ordinate which corresponds to the value of the conditional limit of proportionality.

The method of determining the conditional limit of proportionality is based on the fact that the value of the variable modulus of elasticity at the conditional limit of proportionality is equal to the instantaneous modulus of elasticity reduced by 33%.

According to the equation of deformation diagram in the elastic region of work of material the modulus of elasticity while testing at a constant loading rate is

$$E_{v_\sigma} = \frac{E_{prel}}{1 - k e^{-\alpha \left(\frac{\sigma}{t, v_\sigma}\right)^c}} . \qquad (5.1)$$

Using the condition that the modulus of elasticity at the conditional limit of proportionality is $E_{v_\sigma} = \frac{2}{3} E_{inst}$, we obtain a formula for determining the value of the limit of conditional proportionality:

$$\sigma_{l.p.} = t_1 v_\sigma \sqrt[c]{\frac{1}{\alpha} \ln \frac{2k}{3k-1}} .$$

According to the above formula, at lower loading rates, the limit of proportionality may be reduced to small values close to zero. At higher loading rates, the limit of proportionality may be raised up to the value of ultimate strength. Thus, it is possible to get the values of conditional limit of proportionality at different loading rates, which differ considerably from each other at one and the same allowance in the modulus of elasticity.

The elastic limit for a test at a given loading rate means the stresses corresponding to the point of transition from the elastic region of work of material to the elasto-plastic region on the deformation diagram (see Fig. 27). At the given test rate, the elastic deformation for elasto-visco-plastic materials increases with an increase in stresses at a continuously increasing rate. When passing through the point which corresponds to the elastic limit, besides elastic deformations, plastic deformations begin to form on further increase of load. The law of change in the rate of growth of total deformation at this part differs from the law of change in the rate of growth of the elastic component of deformation. The total deformation grows according to a different law, but the rate of its growth will be more than the rate of growth of the elastic component of total deformation. With a change in loading rate during test, the rate of growth and the values of elastic and total deformations as well as the position of the point corresponding to the elastic limit change.

The method of determination of the elastic limit from the deformation diagram is given below. The underlying idea is as follows. The deformation diagram (Fig. 28) is replotted in terms of the coordinates ε/σ and ε . The diagram of elastic deformations in the new coordinates, at stresses lower than the elastic limit is represented with sufficient accuracy by the straight line 1. The diagram of growth of total deformations (elastic and plastic) at stresses higher than the elastic limit is depicted

in the new coordinates by the straight line 2, but at a different angle of inclination. The new line also conforms to the law of growth of deformations in the elasto-plastic region of work of material. Intersection of the two straight lines permits the determination of the position of the point corresponding to elastic limit on the deformation diagram and on the diagram of limiting elastic deformation.

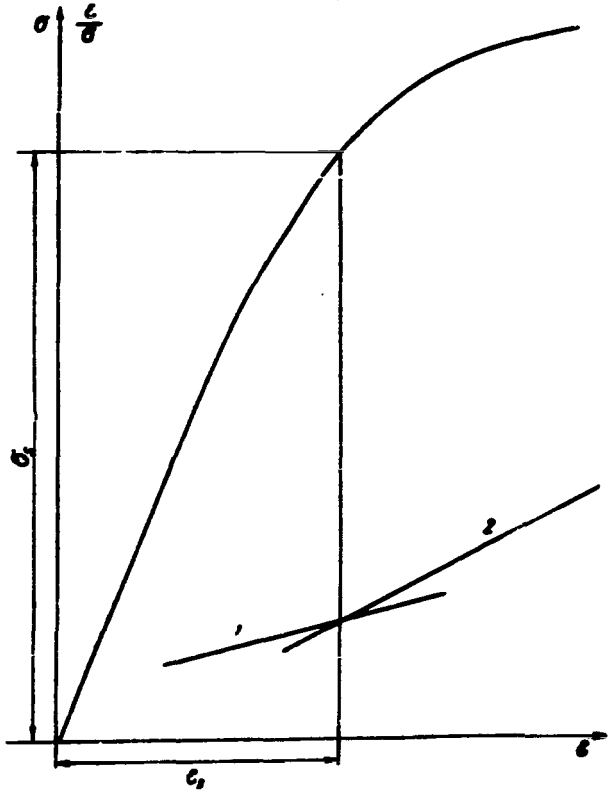

Fig. 28. Determination of elastic limit and limiting elastic deformation from the deformation diagram.

Thus, the determination of the elastic limit and the limiting elastic deformation leads to the linearization of the deformation diagram. The linearization of the deformation diagram done by us showed that this method gives very satisfactory results. The deformation diagram in the case of compression of DSP-plastics and plastic glass and the determination of fracture point by the linearization method are shown in Fig. 29. It is seen that an approximate expression of the equation of the deformation dia-

gram is square parabolic form, as had been assumed at TsNIPS[a/] [38, 40], is correct and it may be used in practice.

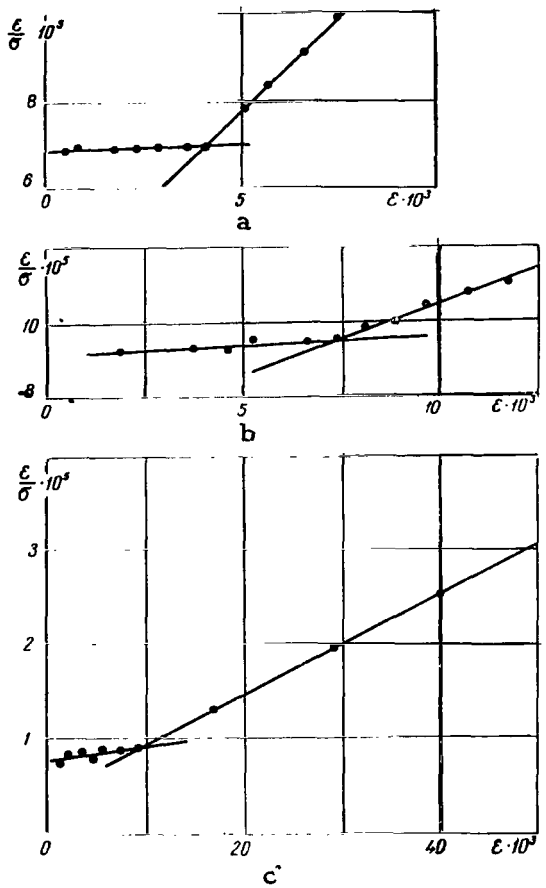

Fig. 29. Linearization of the deformation diagram in the case of compression. a — KAST-V along warp; b — VFT-S along warp; c — DSP-B along the layers of the plies along the fibers of the lining.

The relative total deformation at fracture was determined directly from the total deformation diagram, the instantaneous modulus of elasticity from the instantaneous deformation diagram and the modulus of elasticity from the conditionàl linear part of the total deformation diagram.

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a/ Translator's Note: TsNIPS - Tsentral'nyĭ nauchno-issledovatel'skiĭ institut promyshlenykh sooruzheniĭ (Central Scientific Research Institute of Industrial Structures).

TESTS AT A CONSTANT LOADING RATE

In the test on DSP-plastic and plastic glasses subjected to compression, the diagrams of total, instantaneous elastic, elasto-viscous and plastic deformations were obtained in the process of loading on the basis of the method stated above.

The diagrams of total and instantaneous elastic deformations were obtained directly during tests.

Diagrams of elasto-viscous and plastic deformations were obtained by processing the total deformation diagrams as follows: using the total deformation diagram plotted in the coordinates ε/σ and ε, the linear part of the diagram in the elastic region was extrapolated to the elasto-plastic region of work of material (Fig. 30, a). Such an extrapolation is justified, because the laws of development of elasto-viscous deformations in the elastic and the elasto-plastic region of work of material are similar. Let us draw the diagram of elasto-viscous deformations in the elasto-plastic region of work of material from the extrapolated part of the straight line. For this purpose, we determine the value of the relative elastic deformation at any value of stresses σ and the corresponding value of $\frac{\varepsilon_{elast}}{\sigma}$ (Fig. 30, b). The diagram of elasto-viscous deformations is plotted according to the obtained values of σ and ε_{elast}.

The diagram of plastic deformations was obtained by determining these deformations at each value of stresses as the difference between the total and the elastic (instantaneous elastic and elasto-viscous) deformations. (Fig. 30, c).

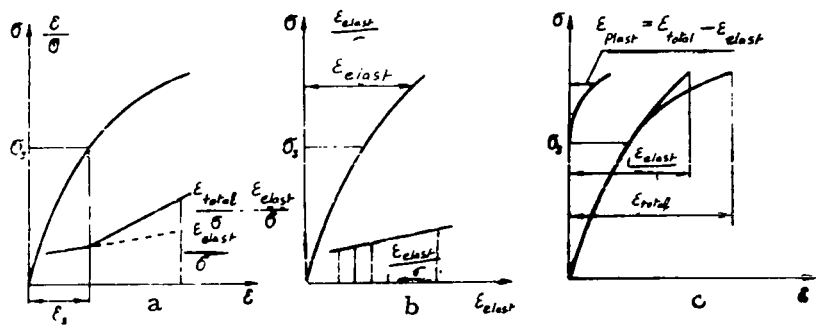


Fig. 30. Graphs for the determination of elasto-viscous and plastic parts of the diagram of deformations.

The values of deformations in the direction of action of force are slightly distorted as a result of their slight increase under the influence of transverse crumpling.

Fig. 31 shows the diagrams of deformations of laminated plastics subjected to compression at a temperature t = 20°C and loading rate of the specimen V_{incr} = 240 dkn/min. The diagram of total, elasto-viscous, elasto-instantaneous and plastic deformations are given and the levels of conditional limit of proportionality and elastic limit are shown here.

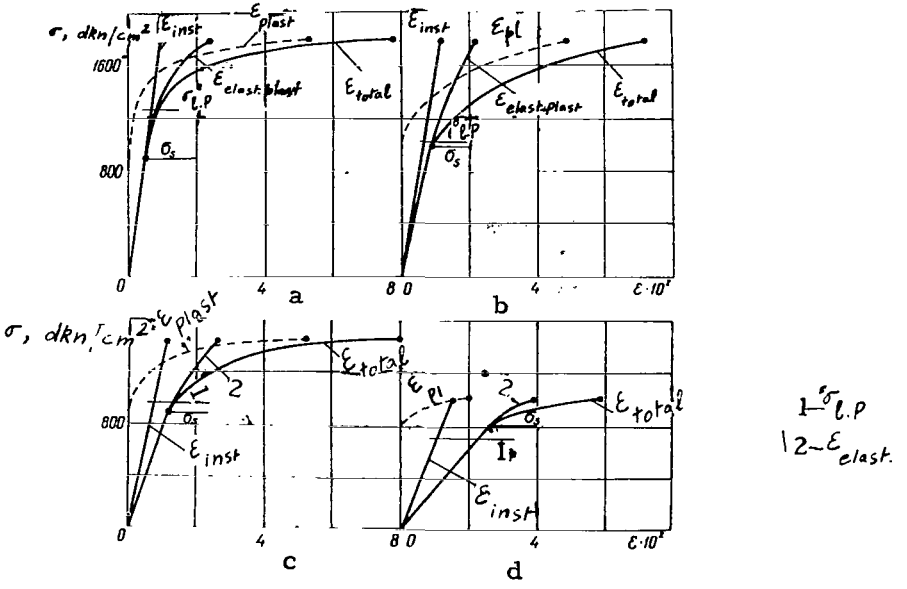


Fig. 31. Deformation diagram in the case of compression along the ply layers along the fibers of the lining of (a) DSP-B plastics (b) DSP-V plastics, (c) DSP-G plastics and (d) transverse to the fibers of lining of DSP-B plastics.

Using the results of prolonged tests, it is possible to determine theoretically the properties of the material by testing at a constant loading rate. Table 8 shows the values of the conditional limit of proportionality for the case of compression, which are calculated by using the formula and the constants obtained in the prolonged test. The values of the conditional limit of proportionality, obtained by processing the experimental deformation diagram of DSP-plastic subjected to compression at a constant rate of increase of stresses V_σ = 170 dkn/cm^2. min and of plastic glasses subjected to compression at a rate V_σ = 2000 dkn/cm^2. min, are also given here. It is seen

Table 8

Material	Direction of action of compressive force	Conditional limit of proportionality, dkn/cm^2	
		exp $\sigma_{l.p.}$	theor $\sigma_{l.p.}$
DSP-B	Along the ply layers along the fibers of lining	1285	1220
	Along the ply layers transverse to the fibers of lining	745	765
	Perpendicular to ply layers	975	1040
DSP-V	Along the ply layers along the fibers of lining	1040	940
	Perpendicular to the ply layers	2400	2500
DSP-G	Along the ply layers along the fibers of the lining	965	985
	Perpendicular to the ply layers	2110	2010
SVAM	Along the ply layers	3010	3060
KAST-V	Along the warp	615	680
	Along the weft	570	540
VFT-S	Along the warp	1390	1320
	Along the weft	800	815

that the experimental values of the conditional limit of proportionality and the values determined by the formula correspond to each other with a sufficient degree of accuracy. Experimental values of conditional limit of proportionality were obtained as the average of ten tests.

ESTIMATION OF INFLUENCE OF LOADING RATE DURING TESTS OF SPECIMENS OF DIFFERENT SIZES

Usually, strength and deformability of a material are estimated by tests of short duration on small specimens.

While calculating the elements of a structure, there arises the question: to what extent do the laboratory data of the investigations on small specimens of the material correspond to the characteristics of the material in large elements of the structure, i.e. what is the influence of the scale factor on the mechanical properties of the material.

Tests of short duration with different rates of loading or deformation are related to some definite period of application of the force on the material. Hence, taking the time factor into consideration, it is possible to explain the influence of the loading or deformation rate on the strength and deformability of a material.

Let us consider the influence of the loading rate on the nature of deformation diagrams.

At a loading rate tending to infinity, elasto-viscous and plastic deformations do not arise in the process of loading, because the time of test tends to zero and only instantaneous elastic deformations appear.

Instantaneous deformations are related to stresses by a linear law, and hence, the deformation diagram has the form of a straight line, the tangent of whose angle of inclination with the deformation axis determines the value of the instantaneous modulus of elasticity.

If the loading rate tends to zero, elasto-viscous deformations develop completely and have a definite value which is related to stresses by a linear law. Deformation diagram in this case also has the form of a straight line, the tangent of whose

angle of inclination with the deformation axis determines the value of the prolonged modulus of elasticity.

If the loading rate is within the range $0 < V_{incr} < \infty$, deformation diagrams are curvilinear, and are situated in the region which is restricted by linear diagrams obtained at the loading rates tending to zero and infinity. All curved diagrams have a common tangent at the origin of the coordinates, which coincides with the rectilinear diagram of instantaneous deformations at the loading rate tending to infinity.

The lower the given constant loading rate, the higher is the deviation of the curvilinear deformation diagram at any of its points from the tangent to it at the origin of the coordinates. Thus, by decreasing the given constant loading rate, the diagram becomes more sloping and is situated lower than deformation diagrams obtained at higher loading rates.

If a constant loading rate is taken for testing specimens with different areas of cross section, the rate of increase in stresses will be different for each of them, and consequently, ultimate strength and relative deformations at the time of fracture will be different.

Thus, in order to obtain one and the same value of ultimate strength and relative elongation at the time of fracture for specimens made of the same material but having different areas of cross section, it is essential that the rate of increase in stresses in these specimens should be the same, and consequently, the rate of increase in load during the test should be proportional to the areas of cross section of the specimens:

$$\frac{V'_{incr}}{F'} = \frac{V''_{incr}}{F''} = const.$$

It is generally considered that in specimens with larger cross sections than the standard ones, the probability of the influence of heterogeneity of material increases.

Conjugate influence of the rate of increase in stresses during tests and heterogeneity of the material results in a decrease in the strength with an increase in the volume of material in the specimens of different sizes. In order to eliminate the influence of the rate of increase in stresses, tests may be conduct-

ed at different rates of increase in load, changing proportionally with the change in the cross section of the specimens. The influence of heterogeneity of the material may also be estimated by such a method of testing. As has been shown by our experiments, the influence of heterogeneity of some materials (DSP-plastic, for example) does not play the same role as generally attributed to it. If the influence of heterogeneity of the material is insignificant, the strength of material at a given constant rate of increase in stresses does not depend on the size of the specimens and changes only with a change in the rate of increase in stresses.

INFLUENCE OF SIZE OF SPECIMENS ON STRENGTH OF DSP-PLASTIC SUBJECTED TO COMPRESSION

A study of the influence of the size of the specimen on the ultimate strength of DSP-plastic was conducted by compressing specimens with a square cross section having the ratio of the sides of the cross section and height as 1 : 1 : 1.5. At this ratio, the ductility of specimens with different cross sections was the same ($\lambda = 5.2$), which practically excludes the loss of stability of the specimens during tests.

According to GOST 5704-51, tests on compression are performed on small specimens having size 15 x 15 x 15 mm at a constant loading rate $v_{incr} = 4500$ dkn/min. At this loading rate, the rate of growth of stresses is $v_{\sigma} = 2000$ dkn/cm^2. min. The latter was taken as constant for tests of specimens of different sizes.

Since anisotropy has an important significance for DSP-plastic, the influence of the size of the specimens on the value of the ultimate strength in the case of compression was studied in three main directions for DSP-B plastic, and in two directions for DSP-V and DSP-G plastics. For the study in each direction, a series of specimens were made consisting of five groups with the ratio of their areas of cross section 1:3:6.5:11:17 for specimens subjected to compression along and transverse to the fibers of lining, and with the ratio 1:1.5:2.7:4.3:7.8 for specimens subjected to compression perpendicular to the ply layers.

The areas of cross section of the specimens were taken within the limits 1.4-25 cm^2.

A number of specimens (10-30) were tested for each type of plastic in each direction of action of force relative to the principal axis of anisotropy and for each area of cross section.

Since the rate of increase in stresses was taken as v_σ=2000 dkn/cm^2. min, tests were conducted at different rates of increase in load within the limits v_{incr}= 2900 to 5300 dkn/min depending on the area of cross section of the specimens.

Tests were conducted on hydraulic machines of the type "Emery Baldwin" having a capacity of 30, 100 and 300 tons at a controlled loading rate. In all, about 600 specimens having different areas of cross section were tested. The results of the tests are presented by curves (Fig. 32) plotted in the following coordinates: average value of ultimate strength, for each series of specimens of the given size, as the ordinate and the ratio F_n/F_a of the average values of the areas of cross section of each series of specimens to the average value of the area of the series

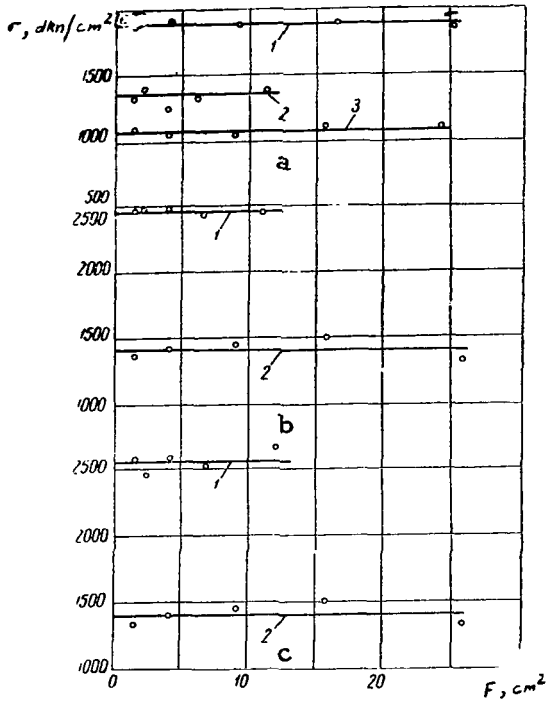


Fig. 32. Effect of the size of the specimens on ultimate strength in the case of compression of:

a - DSP-B (1) along the ply layers along the fibers of lining, (2) perpendicular to the ply layers, (3) transverse to the fibers of lining;

b - DSP-V (1) perpendicular to the ply layers, (2) along the fibers of lining;

c - DSP-G (1) perpendicular to the ply layers, (2) along the ply layers along the fibers of lining.

of specimens with minimum cross section as the abscissa. The data for plotting the curves and the results of statistical processing are presented in Table 9. As seen from the curves and Table 9, the strength of DSP-plastic does not depend on the area of cross section of the specimens at the given constant rate of increase in stresses. Thus, in spite of considerable macroheterogeneity of DSP-plastic, the effect of heterogeneity on strength of specimens of different sizes appears to be insignificant and may be neglected in practice.

Following the existing view that the scale factor is determined by the degree of heterogeneity of material of the specimen which is determined by its size, the influence of the rate of increase in stresses is not taken into consideration while determining the ultimate strength of the material. The conducted experiments lead us to conclude that, while estimating strength of a material having appreciable volume, it is essential to consider the time of the test. This consideration makes it possible to explain more accurately the influence of the heterogeneity of the material.

PREREQUISITES OF RAPID METHODS FOR DETERMINING THE PROLONGED CHARACTERISTICS OF STRENGTH AND DEFORMABILITY

Finding rapid methods presents considerable scientific and practical interest because they make it possible to determine the characteristics of prolonged strength and deformability of a material in the course of a short time interval. Since a rapid method can give the characteristics with lesser or higher degree of accuracy, a comparison of the results of the usual prolonged tests with the results of rapid methods is essential.

Deterimination of the prolonged characteristics of rapid methods is based on the utilization of the data of short-duration tests at constant rates of loading or deformation. The essential point is that there exists a linear relation between the elastic limit σ_s and the limiting elastic deformation ε_s, and that the relation between elastic limit σ_s and the rate of development of plastic deformation is conditionally linear in the stress interval from $\sigma_s = \sigma_{l.l.r.}$ to $\sigma_s = 0.8\,\sigma_{str}^{lim}$.

The experimentally obtained diagrams showing the relation between elastic limit σ_s and the rate of development of plastic

Table 9

Material	Direction of action of compressive force	Ratio of the area of the specimen to the maximum area of the section	Ultimate strength, dkn/cm²			Ultimate strength (mean for a series of specimens), dkn/cm²	Results of statistical data processing	
			Maximum	Minimum	Average		Average quadratic deviation	Average arithmatic mean deviation
DSP-B	Along the ply layers along the fibers of lining	1.0	2000	1730	1877	1883	131	34
		2.9	2090	1745	1916			
		6.5	2110	1705	1891			
		11.5	2060	1800	1903			
		17.5	1870	1765	1832			
	Along the ply layers transverse to the fibers	1.0	1195	1020	1109	1090	60	16
		2.8	1074	1010	1049			
		6.3	1094	1010	1049			
		11.0	1155	1080	1139			
		16.9	1135	1030	1106			
	Perpendicular to the ply layers	1.0	1418	1269	1321	1342	86	22
		1.5	1504	1390	1431			
		2.7	1353	1160	1236			
		4.3	1361	1305	1332			
		7.8	1426	1370	1391			
DSP-V	Along the ply layers along the fibers of lining	1.0	1391	1240	1320	1406	114	30
		2.8	1485	1215	1406			
		6.3	1590	1385	1464			
		10.9	1560	1455	1525			
		17.9	1350	1285	1314			
	Perpendicular to the ply layers	1.0	2380	2050	2460	2452	304	78
		1.4	2830	2060	2463			
		2.7	2895	2140	2471			
		4.5	2490	2375	2424			
		7.7	2810	2080	2440			
DSP-G	Along the ply layers along the fibers of lining	1.0	1500	1290	1380	1402	113	29
		2.8	1570	1270	1412			
		6.3	1435	1287	1370			
		10.9	1585	1290	1458			
		17.9	1485	1225	1389			
	Perpendicular to the ply layers	1.0	2745	2350	2568	2563	206	53
		1.6	2582	2429	2430			
		2.8	2764	2186	2589			
		4.7	2780	2220	2530			
		8.3	2860	2410	2700			

deformations v_s for the case of sustained constant loads have been presented earlier and it has been established that this relation is curvilinear, particularly at stresses approximately equal to the value of ultimate strength. Considering the right hand part of the curves, we notice that at stresses approximately equal to the limit of long-term resistance, the relation between σ_s and v_s may be practically expressed by a straight line. The stress interval, within the limit of which the approximation of the curvilinear relation between σ_s and v_s to a rectilinear relation is possible, is, according to our calculations, from $\sigma_s = \sigma_{l.l.r}$ to $\sigma_s = 0.80\, \sigma_{str}^{lim}$, which permits us to accept the linear relation between σ_s and v_s within a definite stress range without appreciable errors.

The second fairly important rule for establishing and developing rapid methods of determination of the prolonged characteristics of strength and deformability is the relation between elastic limit σ_s and limiting elastic deformation ε_s. As mentioned earlier, this relation is linear.

If the linear relation between σ_s and v_s is approximate and is correct within a definite interval of stresses, the linear relation between σ_s and ε_s is true in the entire range from $\sigma = \sigma_{l.l.r.}$ and $\varepsilon_s = \frac{\sigma_{l.l.r.}}{E_{prol}}$ to $\sigma_s = \sigma_{str}^{lim}$ and $\varepsilon_s = \frac{\sigma_{str}^{lim}}{E_{inst}}$. This fact has been observed in our experiments and also in the experiments of other research workers [18, 20, 36].

The abovementioned linear relations between σ_s and v_s, σ_s and ε_s are true for sustained constant load, as well as for tests of short duration. Simultaneous attainment of the limiting values of stresses and deformations in different methods of loading is one of the fundamental assumptions. This assumption was verified and confirmed by us experimentally. Experimental verifications of rapid methods of determination of the prolonged characteristics presented below testify to its correctness since these rapid methods are in fact based on this assumption.

Thus, the task of determining the prolonged characteristics reduces to finding out the linear relations between σ_s and ε_s, σ_s and v_s from tests of short duration.

RAPID METHOD FOR DETERMINING THE LIMIT OF LONG-TERM RESISTANCE

The most reliable method for determining the limit of long-term resistance consists in plotting the relation between load and time till fracture on the basis of the experimental data of the ultimate strength, obtained under the action of constant loads of different magnitudes. Then the experimental data are processed by statistical methods and the minimum value of the breaking load which corresponds to the limit of long-term resistance is established. This method needs a large number of homogeneous specimens, special machines permitting constancy of the applied loads, observation of constant hydrodynamic conditions during the experiments, and above all, sufficient time for conducting the experiments. All these facts forced us to look for some rapid method for determining the limit of long-term resistance.

A few rapid methods have been proposed so far [5, 20, 36, 38, 59, 63, 69]. The most rational and experimentally confirmed methods are those suggested by the Institute of Mechanics of the Academy of Sciences of the Ukranian SSR [5] and Central Scientific Research Institute of Industrial Construction [69].

The first method is based on the extrapolation of the conditional linear relation between the rate of development of plastic deformations and continuously acting stresses. It gives the value of the limit of long-term resistance as the elastic limit at zero rate of development of plastic deformation. The good agreement of the limit of long-term resistance obtained by this method with its value from the prolonged resistance curve permits us to consider this method as completely reliable. However, it is necessary to mention that experiments have to be conducted for several days in order to determine the rate of development of plastic deformations by this method.

The second method is based on the extrapolation of the curve showing the relation between ultimate strength and the rate of loading during the test. The determination of the limit of long-term resistance by this method needs a large number of specimens and is less accurate because the curve is extrapolated.

Thus, so far there is no sufficiently simple quick method, which would permit us to obtain without much expenditure of

time and specimens and with the use of ordinary test machines, the characteristics of prolonged strength, i.e., limit of term long-term resistance.

We have worked out a rapid method for determining the limit of long-term resistance which needs minimum time and may be performed on ordinary test machines.

As stated earlier, under the action of a constant load giving rise to stresses higher than the limit of long-term resistance, there initially arise instantaneous elastic deformations. Then, elasto-viscous deformations develop which attain a limiting value corresponding to the given stresses. After this stage, the growth in elastic deformations with time does not take place, but only plastic deformations occur with a constant rate which is equal to the limiting rate of development of elastic deformations at the end of the limiting time interval. Different continuously acting stresses correspond to different values of limiting elastic deformation and different rates of their development. The relations between the stresses and the value of limiting elastic deformation and its rate of increase at stresses 20-30% higher than the limit of long-term resistance are expressed by straight lines and have been confirmed by repeated experiments. Specifically, one of these relations, namely, the relation between stresses and limiting rate of increase in elastic deformation, equal to the rate of increase in plastic deformation, is the basis of the rapid method of IM AN UkSSR [10].

Thus, the formation of plastic deformations due to a sustained load occurs after the simultaneous attainment of the limiting values by stresses and elastic deformations.

In tests of short duration at a given rate, it is also possible to differentiate two regions -- the region of elastic deformations and the region of plastic deformations. In the first region, deformations initially grow according to a linear law and on approaching the limit of elastic region, i.e., the elastic limit or the plastic flow limit, according to the terminology of Yu. M. Ivanov, they follow a curvilinear law. The transition to the second region is characterized by the change of curvature on the deformation diagram.

The transition from one region of work of material to another is characterized not only by the attainment of the limiting

value of stresses, but also by the attainment of limiting elastic deformation corresponding to these stresses, i.e., the formation of plastic deformations is possible only after the simultaneous attainment of the limiting values of stresses and elastic deformations.

This permits us to establish that, in the case of tests at a given rate, corresponding to the elastic limit, there is a limiting rate of development of elastic deformations, which is equal to the rate of development of plastic deformations, provided the stresses are maintained constant subsequently, i.e., we come to the case of the constant sustained load.

The proposed rapid method for determining the limit of long-term resistance is based on establishing the relation between elastic limit and the limiting rate of development of elastic deformations from tests at different constant loading rates.

Thus, the problem is to find out the linear relation, which is the basis of the rapid method of IM AN UkSSR, and which can be determined from prolonged tests and tests of short duration with constant but equal rates of loading.

For this purpose, we draw deformation diagrams (Fig. 33) at three or four different rates of loading, from which we obtain the elastic limit (plastic flow limit) σ_s by the known methods, for example, by the method of TsNIPS or by the abovementioned methods of linearizing the deformation diagram. Next, we replot the deformation diagrams in terms of the coordinates: deformation - testing time (Fig. 34), where testing time is defined as the ratio of the stresses to the rate of increase in stresses and then, we find the rate of growth of the limiting elastic deformations v_s corresponding to the obtained elastic limits. After this, we plot the curve showing the relation between stresses σ_s and rate v_s, process the data statistically and select the equation of the straight line and by extrapolating along the ordinate, we find the value of limit of long-term resistance as the stresses at deformation rate $v_s \to 0$, which corresponds to the loading rate equal to zero.

The results of experimental verification of the proposed method and its comparison with the method of IM AN UkSSR are given below [5]. The verification was conducted on wood, DSP-plastic and plastic glass. The results of the experiments for determining the relation between stresses and the rate of plastic

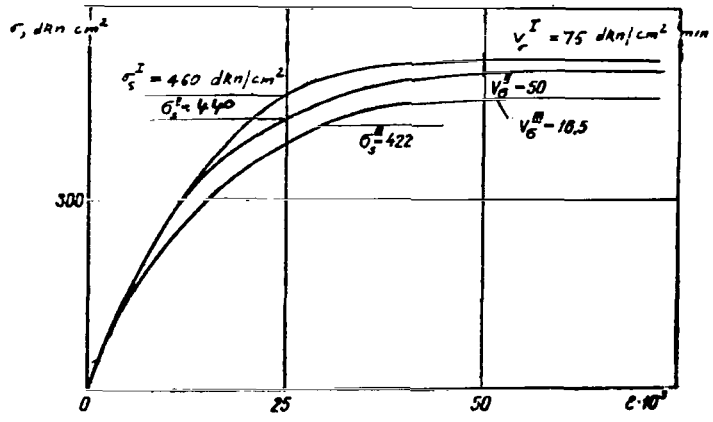


Fig. 33. Deformation diagram of pine wood subjected to compression along the fibers.

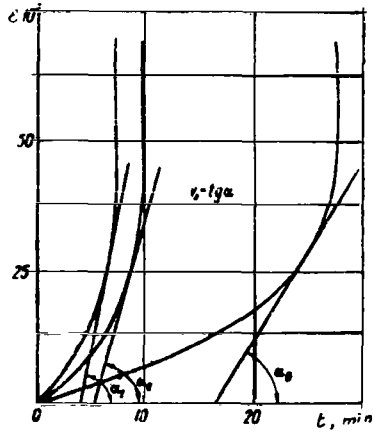


Fig. 34. Deformation diagrams of pine wood at compression in the coordinates ε, t.

deformation of pine wood subjected to compression in the case of a continuous action of load (points) are presented in Fig. 35. It also shows the straight line, selected by the least square method, extrapolation of which gives the value of the limit of long-term resistance according to rapid method of IM AN UkSSR. The results of the determination of the relation between stresses and limiting rate of growth of elastic deformations, obtained from tests at different loading rates (squares) are also presented here. Analogous curves for DSP-G plastic subjected to compression along the ply layers along the fibers of lining are given in Fig. 36.

The curves presented here show that the proposed rapid method is completely reliable and also confirms the presence

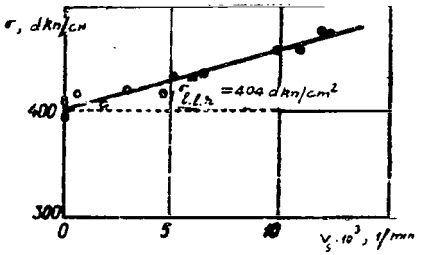


Fig. 35. Determination of limit of long-term resistance by ordinary method and rapid method in the case of pine wood subjected to compression along fibers.

of a general critical state of the material for prolonged tests and tests of short duration.

RAPID METHOD FOR DETERMINING THE PROLONGED MODULUS OF ELASTICITY

The determination of prolonged modulus of elasticity from the curves showing the development of deformation with time under the action of a constant load needs tests for a prolonged period (a month and more). Since this characteristic is very important, the question arose for working out a rapid method for its determination.

A method is proposed here for obtaining the prolonged modulus of elasticity from the results of tests of short duration, based on the utilization of the linear relation between elastic limit and limiting elastic deformation. This relation has the form of a straight line within the limits, which in terms of stresses correspond to the limit of long-term resistance $\sigma_{l.l.r.}$ and the limit of ultimate strength σ_{str}^{lim}. In terms of deformations, these limits correspond to prolonged limiting deformation at $\sigma = \sigma_{l.l.r.}$ and limiting instantaneous deformation at $\sigma = \sigma_{str}^{lim}$

The corresponding deformation in both the cases may be obtained as the elastic deformation at the time of fracture:

$$\varepsilon_{prol} = \frac{\sigma_{l.l.r.}}{E_{prol}} \quad , \quad \varepsilon_{inst} = \frac{\sigma_{str}^{lim}}{E_{inst}} \quad ,$$

where E_{prol} and E_{inst} are prolonged and instantaneous moduli of elasticity.

Prolonged and instantaneous moduli of elasticity may be obtained from the given expressions as the ratio of the corresponding stresses to deformations:

$$E_{prol} = \frac{\sigma_{l.l.n.}}{\varepsilon_{prol}}, \quad E_{inst} = \frac{\sigma^{lim}_{str}}{\varepsilon_{inst}}.$$

Fig. 36 shows (on the left hand side) the rectilinear relation between stresses σ_s and the limiting elastic deformations ε_s. Using the right hand part of this graph for determining the limit of long-term resistance and knowing the data from the deformation diagrams for plotting the left hand part, it is possible to find out the limit of long-term resistance and prolonged deformation by extrapolation and to determine the prolonged modulus of elasticity from the value of prolonged deformation.

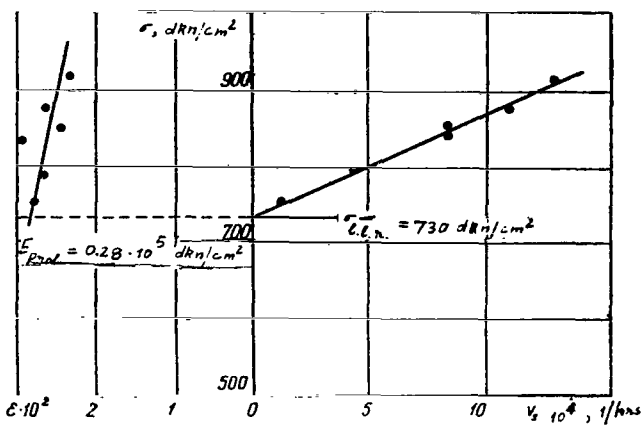


Fig. 36. Determination of limit of long-term resistance and prolonged modulus of elasticity by the quick method in the case of DSP-G plastic subjected to compression along the ply layers along the fibers of lining.

Thus, in order to determine prolonged modulus of elasticity, three or four deformation diagrams are drawn for tests of short duration with different rates of loading or deformation and the points,delimiting the elastic and elasto-plastic regions of work of material are obtained. These points determine the values of elastic limits, the corresponding limiting elastic deformations and the rate of growth of plastic deformations.

Using the obtained values, we plot the relation between stresses and the rates of growth of plastic deformations (right

hand part of Fig. 36), which forms the basis of the rapid method for determining the limit of long-term resistance. Then, using the obtained values of elastic limits and corresponding values of limiting elastic deformations and taking the linear relation as the basis, we plot the left hand part of the curve.

Extrapolating the obtained relation to stresses equal to the limit of long-term resistance, the corresponding limiting prolonged elastic deformation is obtained. Prolonged modulus of elasticity is determined as the ratio of the limit of long-term resistance to prolonged elastic deformation.

Experimental verification of the proposed method, conducted on pine wood with 10% moisture, DSP-plastic and KAST-V plastic glass showed satisfactory agreement between the values of prolonged characteristics, obtained by the abovementioned method and those obtained from the creep curves by ordinary prolonged tests of material at constant loading. Table 10 presents a comparison of the values of limit of long-term resistance and prolonged modulus of elasticity for pine wood, DSP-plastic and KAST-V plastic glass subjected to compression obtained by the rapid method and by prolonged tests. Good agreement of

Table 10

Material	Direction of action of compressive force	Limit of long-term resistance $\tau_{l.n.}$, dkn/cm^2		Prolonged modulus of elasticity $E \times 10^{-5}$, dkn/cm^2		Instantaneous modulus of elasticity $E \times 10^{-5}$, dkn/cm^2	
		Prolonged tests	Rapid method	Prolonged tests	Rapid method	Stepped steps	From graph
Pine wood	Along fibers	332	320	0.78	0.66	1.63	1.85
DSP-B	Along the ply layers along the fibers of lining	735	755	0.15	0.18	0.48	0.52
DSP-V	Along the ply layers along the fibers of lining	750	767	0.26	0.25	1.06	1.23
DSP-G	-do-	740	730	0.25	0.28	0.80	0.93
KAST-V	Along warp	617	635	0.73	0.85	1.34	1.56

the experimental values permits us to consider that the proposed rapid methods are effective.

Using the linear relation between σ_s and ε_s, it is possible to obtain the value of instantaneous modulus of elasticity. For this, it is necessary to obtain from tests of short duration the limit of ultimate strength as the asymptote of the curve showing the relation between ultimate strength and loading rate, and to obtain the corresponding instantaneous elastic deformation by extrapolating the linear relation between σ_s and ε_s to stresses $\sigma_s = \sigma_{str}^{lim}$. Then, taking the ratio of the limit of ultimate strength and this instantaneous deformation, the instantaneous modulus of elasticity can be calculated. The value of instantaneous modulus of elasticity for pine wood, DSP-plastic and plastic glasses, obtained by this method and a comparison with the values obtained by stepped tests are presented in Table 10. Good agreement of the values shows that the proposed method may be used for the determination of the instantaneous modulus of elasticity.

CHAPTER VI

INVESTIGATIONS ON STRENGTH AND DEFORMABILITY OF LAMINATED PLASTICS AND WOOD SUBJECTED TO BENDING

THEORY OF LIMITING STATE OF BAR SUBJECTED TO BENDING

The theory of limiting state of bars subjected to bending has been worked out most elaborately for wood. Investigations on the limiting state of plastic glasses and laminated plastics subjected to bending are comparatively few. Since all these materials belong to one class, many theoretical studies carried out for wood can be extended to synthetic materials.

The theory of bending is based on the hypothesis of planar sections, the experimental verification of which indicates the possibility of its application [7, 12, 20, 27, 36, 44].

The theory of bending of wood has been fully reflected in the works of F.P. Belyankin [6, 7, 8], A.I. Rabinovitch [72], A.A. Dubrovin [25], V.P. Pinadzhan [70], A.B. Gubenko [22, 23], A. Yu. Romashevskiĭ [85] and V.N. Bykovskiĭ [20].

Without considering the influence of the time factor on the diagram of normal stresses and assuming that visco-elasto-plastic material works in the stretched zone of the section of the bar elastically till fracture of the end fibers, and elasto-plastically in the compressed zone, F.P. Belyankin [6, 7] and also independently Prager proposed triangular-trapezoidal diagram of distribution of normal stresses along the section in the limiting state for the case of bending. This diagram was the basis of the investigations by other authors.

Using ordinary methods of strength of materials, F.P. Belyankin found the expression for bending moment at limiting state as

$$M_{lim} = \mu \sigma_{compr} W , \qquad (6.1)$$

where σ_{compr} is the ultimate strength for compression; W is the moment of resistance of the section and μ is the coefficient of increase in the carrying capacity due to the formation of plastic deformations in the compressed zone of the bar, which is dependent on the ratio of ultimate strengths for compression and tension.

Since, in formula (6.1), the influence of the time factor was not considered and hence the theoretical values of limiting bending moments differed from the experimental ones, introduction of an additional coefficient was necessary, which indirectly takes into consideration the time factor; it was termed by F.P. Belyankin as the coefficient of supporting action, which, according to his assumptions, was determined either theoretically or experimentally [6, 7]. It should be mentioned that the time factor was also not taken into consideration in the publications [6, 7].

Investigations of the Soviet scientists on further development of the theory of limiting state of bending were mainly aimed at obtaining partial solutions and experimental verification of the theory. Works of A.I. Rabinovich [72], A.B. Gubenko [23] and Ya. T. Panovko [67] were devoted to the problems of determination of deformations beyond the limit of proportionality. These also did not consider the influence of time factor on the deformability during bending. In order to investigate the strength and deformability during bending, deformation diagrams for the case of compression and tension, obtained by tests of short duration were used. However, as has been mentioned in the works of V.N. Bykovskiĭ [20] and S.E. Fraĭfeld [89], the use of deformation diagrams as the diagram of normal stresses is not correct in principle, because deformation diagrams at each moment of time depend on the rate of test.

INFLUENCE OF TIME FACTOR ON STRENGTH AND DEFORMABILITY DURING BENDING

The most complete data on prolonged strength in the case of bending has been obtained for wood [4, 9, 10, 12, 36]. Laminated plastics are much less studied. In recent years, the volume of experimental works is increasing [51, 57, 68, 77]. As shown by the experiments, the nature of prolonged strength curvess in the case of bending is the same as that in the case of tension and compression. This permits to establish a relation between changes in strength with time during bending, tension and compression.

Considerable amount of work has been done on the study of creep during bending. The more important are the works of F.P. Belyankin [5, 10], Yu. M. Ivanov [38, 39], A.M.Ivanov [36], V.N. Bykovskiĭ [18, 20], V.P. Kotsegubov [54], K.P. Kashkarov [49], V.G. Lennov [57], N.L. Leont'ev [58] and B.A. Kiselev [51]. However, it should be noted that majority of the works are devoted to experimental studies of creep during bending. The problems of the theory of bending with consideration of the time factor have not been sufficiently worked out, particularly for the elasto-plastic region of work of the material Investigations of V.P. Kotsegubov [54] and V.G. Lenov [57] are mainly devoted the elastic region of work of the material. Calculations of plastic deformations and problems of limiting carrying capacity in the case of bending by taking the time factor into consideration were studied theoretically by B.N. Bykovskiĭ [20], A.M. Ivanov [36], V.G. Lenov [57] and other authors, but a complete solution of this problem does not exist in literature.

PRINCIPAL CASES OF WORKING OF BAR UNDER THE ACTION OF A SUSTAINED CONSTANT BENDING LOAD

It is known from experiments that a constant bending load, depending on its value, may or may not cause fracture of a bar in the course of the final time interval. Thus, there exists a limiting value of bending moment, under the action of which the breaking of the beam takes place during an infinitely long period of time.

On bending a bar, one part of the material is subjected to compression and the second part to tension. Depending upon the value of the stresses caused by a continuously acting bending moment, the bar may undergo deformation elastically or elasto-plastically in the compressed or stretched zones. In order to determine the type of deformation, it is necessary to know the minimum values of ultimate strength and the limit of long-term resistance of the material under compression and tension.

For DSP-plastics, plastic glasses and wood, the limit of ultimate strength σ_{str}^{lim}, elastic limit σ_s and limit of long-term resistance $\sigma_{l.t.r.}$ are higher in the case of tension than those in the case of compression. Different values of σ_{str}^{lim}, σ_s

and $\sigma_{l.l.n.}$, for compression and tension lead to the fact that during bending the development of axial and total deformations with time takes place according to different rules, depending upon the value of the bending moment.

Let us assume, that the planar section hypothesis is true in the case of prolonged bending, that the moduli of elasticity for compression and tension are equal at any time interval of action of constant load and that the fracture of the bar subjected to bending is determined by the rupture of the stretched fibers.

In the case of instantaneous loading, stresses develop instantaneously in the compressed and stretched zones of the bar. Here stresses are determined by Nawe's formula

$$\sigma = \frac{M}{W},$$

where M is the continuously acting bending moment and W is the moment of resistance of the section of the bar.

Determination of stresses by Nawe's formula is possible, if the bending moment M is less than or equal to $\sigma^{lim}_{str.compr} W$. But if the bending is higher than $M = \sigma^{lim}_{str.compr} W$, the stresses are determined from the trigonal-trapezoidal diagram of normal stresses [7, 93, 94] provided that the redistribution of stresses along the height of the section takes place sufficiently quickly.

On bending the bar by a continuously acting load, four cases of deformation with time, differing from each other, are possible, depending upon the value of the load.

Case 1. The stresses, applied to the compressed end fiber are less than or equal to the limit of long-term resistance in the case of compression, and the stresses applied to the stretched end fiber are less than the limit of long-term resistance in the case of tension:

$$\left.\begin{aligned} \sigma_{compr} &\leq \sigma_{l.l.n.compr}, \\ \sigma_{tens} &< \sigma_{l.l.n.tens}. \end{aligned}\right\} \qquad (6.2)$$

Correspondingly, the deformations of the end fibers during an infinitely long time of action of the bending moment is

$$\left.\begin{aligned} \varepsilon_{compr} &\leq \frac{\sigma_{l.l.r.\,compr}}{E_{prol}}, \\ \varepsilon_{tens} &< \frac{\sigma_{l.l.r.\,tens}}{E_{prol}}, \end{aligned}\right\} \tag{6.3}$$

when E_{prol} is the prolonged modulus of elasticity.

The continuously acting bending moment must be less than or equal to the limiting bending moment, at which only the elastic work of material of the bar is possible:

$$M \leq M_{lim.\,elast}. \tag{6.4}$$

Accordingly, the maximum sag must be less than or equal to the limiting sag at an infinitely long time of action of the bending moment:

$$f \leq f_{prol.\,elast} \tag{6.5}$$

In the first case, elastic and elasto-viscous deformations of the fibers, but not plastic deformations, arise in the compressed and stretched zones of the bar. Elastic deformations develop for infinitely long time, without causing fracture of the bar. Total deformations (sag) also consist of instantaneous elastic and elasto-viscous deformations, which develop with time and do not lead to fracture at an infinitely long time of action of the constant bending moment.

Case 2. The stresses in the compressed end fiber are greater than the limit of long-term resistance in the case of compression, but the stresses in the stretched end fiber are less than the limit of long-term resistance in the case of tension:

$$\left.\begin{aligned} &\sigma_{compr} > \sigma_{l.l.r.\,compr}, \\ &\text{и} \\ &\sigma_{tens} < \sigma_{l.l.r.\,tens} \end{aligned}\right\} \tag{6.6}$$

Deformation of the stretched end fiber is

$$\varepsilon_{tens} < \frac{\sigma_{l.l.r.\,tens}}{E_{prol}}; \tag{6.7}$$

the continuously acting bending moment must be higher than the limiting elastic moment, but less than the bending moment which causes the failure of the bar at infinitely long time of action:

$$M_{lim.\,elast} < M < M_{lim.\,prol}, \tag{6.8}$$

and the sag at infinitely long time of action of the bending moment must be higher than the limiting elastic one, but less than the sag, which is caused by the bending moment $M_{lim.\,prol}$

$$f_{prol.\,elast} < f < f_{lim.\,prol} \tag{6.9}$$

In the second case, instantaneous elastic and elasto-viscous deformations, which develop with time, are formed in the compressed and the stretched zones of the bar and plastic fibrous deformations also develop with time in the compressed zone. In spite of the formation of plastic deformations in the compressed zone total deformations (sags) develop for infinitely long time but do not lead to the fracture of the bar, since its breaking is caused by the rupture of the stretched fibers and in this case they work elastically.

At constant value of the bending moment, instantaneous elastic sags are formed initially, and the elasto-viscous sags develop with time. The latter develop in the course of the time interval equal to the limiting value for the compressed end fiber, after which the development of elasto-viscous plastic sag takes place for an infinitely long time without fracture of the beam.

Case 3. The stresses in the compressed end fiber are higher than the limiting of long-term resistance in the case of compression and the stresses in the stretched end fiber are higher than the limit of long-term resistance in the case of tension:

$$\left.\begin{aligned} \sigma_{compr} &> \sigma_{l.l.r.\,compr}, \\ \sigma_{tens} &> \sigma_{l.l.r.\,tens} \end{aligned}\right\} \tag{6.10}$$

Deformation of the stretched end fiber is

$$\varepsilon_{tens} > \frac{\sigma_{l.l.r.\,tens}}{E_{prol}}; \tag{6.11}$$

the continuously acting bending moment will be higher than the bending moment, which causes fracture of the beam at infinitely long time of action:

$$M > M_{lim.prol}, \tag{6.12}$$

and the sag will be higher than the limiting elasto-visco-plastic value:

$$f > f_{lim.prol} \tag{6.13}$$

In the third case, fibrous deformations develop similar to those in the second case, and also plastic fibrous deformations in the stretched zone of the bar. Here instantaneous elastic sag is formed in the beginning, and then elasto-viscous deformation develops in the compressed end fiber during the limiting time interval (the first limiting time interval) followed by development of elasto-visco-plastic sag. The latter develops with time till plastic deformations arise in the stretched zone of the bar, i. e., the time interval of its development (the second limiting time interval) depends upon the limiting time interval of the development of elasto-viscous deformations of the stretched end fiber. After the complete development of elasto-visco-plastic sag, the development of plastic sag takes place, which leads to the fracture of the bar in the course of some time interval.

If the stresses in the stretched end fiber, its fibrous deformation, bending moment and sag do not exceed the values according to (6.9) to (6.13), but are equal to them, the fracture of the bar will occur in an infinitely long time of action of the load without the formation of plastic deformations in the stretched zone of the bar, i. e., due to infinitely prolonged development of elasto-visco-plastic sag.

Correlation of stresses in the compressed and stretched end fibers should be such, that plastic deformations arise initially in the stretched zone and then in the compressed zone of the bar, i. e., limiting elastic deformation is reached in the stretched end fiber.

Case 4. Fracture of the bar due to the rupture of the stretched fibers is also possible without the formation of plastic deformations in the compressed zone, if the material in the

section is distributed mainly in its compressed part. Stresses in the compressed end fiber are

$$\sigma_{compr} < \sigma_{l.l.n.\,compr}, \tag{6.14}$$

and stresses in the stretched end fiber are

$$\sigma_{tens} > \sigma_{l.l.n.\,tens} \tag{6.15}$$

The criteria of fibrous deformations in the stretched zone of total deformations and of bending moment in this case are the same as in the third case. Here, instantaneous elastic deformation is formed initially and then elasto-viscous deformations develop in the stretched and compressed zones of the bar. If the stresses in the stretched zone are higher than in the compressed zone then after attaining the limiting elastic deformation in the stretched end fiber, plastic deformations in the stretched zone of the bar develop till its fracture whereas the compressed fibers work elastically.

CHAPTER VII

CREEP DUE TO BENDING IN ELASTIC REGION OF WORK OF MATERIAL

DEFORMABILITY IN THE CASE OF BENDING BY A CONSTANT PROLONGED ACTING LOAD

Let us assume that in the formation of elastic deformations in a bent bar, the rule of planar sections and the proportionality rule of axial stresses and axial instantaneous elastic deformations hold good. Then, on practically instantaneous loading, the cross sections, turning instantaneously, remain planar and the law of distribution of normal stresses along the height of the section will be linear. After loading, the load applied on the bar remains constant and gives rise to stresses, constant in their values and distributed along the height of the section according to a linear rule. Since under the axial action of a load, the value and the law of change in tensile and compressive moduli of elasticity with time are practically the same, and the law of proportionality between stresses and deformations is valid for any given time interval, the elasto-viscous fibrous deformations due to bending of the bar, which develop with time, cause a change in the angle of deflection of the cross section of the bar, the cross section itself remaining planar at any time interval.

The angle of deflection is determined by the conventional methods of strength of materials. For a bar of rectangular cross section, for example, the instantaneous angle of deflection of the cross section may be determined by using the graphical analytical method:

$$\left(\theta_{inst}\right)_x = \frac{R_f - F_x}{IE_{inst}}, \qquad (7.1)$$

where R_f is the fictitous support reaction; F_x is the area of the moment diagram from the origin of the coordinates to the given section at a distance x; E_{inst} is the instantaneous modulus of elasticity and I is the moment of inertia of the section.

In the case of bending, the value of the instantaneous curvature of the bar at the time of formation of instantaneous elastic fibrous deformations is

$$\frac{1}{\rho_{inst}} = \frac{\varepsilon_1 + \varepsilon_2}{h}, \tag{7.2}$$

where ε_1 is the instantaneous elastic deformation of the stretched end fiber; ε_2 is the instantaneous elastic deformation of the compressed end fiber; and h is the height of the cross section of the bar.

For a bar of rectangular cross section

$$\left.\begin{aligned} \varepsilon_1 = \varepsilon_2 = \varepsilon_{inst} = \frac{\sigma}{E_{inst}}, \\ \frac{1}{\rho_{inst}} = \frac{2\varepsilon_{inst}}{h} = \frac{2\sigma}{hE_{inst}} \end{aligned}\right\} \tag{7.3}$$

Here, σ denotes the absolute value of the stresses in the end fibers.

Since the modulus of elasticity varying with time in the sustained tests is

$$E_t = \frac{E_{prol}}{1 - ke^{-\alpha\left(\frac{t}{t_1}\right)^c}},$$

the curvature at the end of the time interval t will be

$$\frac{1}{\rho_{total}} = \frac{2\sigma}{hE_t},$$

or

$$\frac{1}{\rho_{total}} = \frac{2\sigma}{hE_{prol}}\left(1 - ke^{-\alpha\left(\frac{t}{t_1}\right)^c}\right) \tag{7.4}$$

According to the bending moment diagram, the stresses in the end fiber, depending upon the position of the section, are determined by

$$\sigma = \frac{M_x}{W}.$$

Putting the value of σ in the equation (7.4), we get

$$\frac{1}{\rho_{total}} = \frac{M_x}{IE_t}. \tag{7.5}$$

Deformation of any fiber in the given section of the bar, situated at a distance z from the neutral axis is

$$(\varepsilon_t)_z = \frac{(\varepsilon_t)_{z=\frac{h}{2}}\, 2z}{h}, \qquad (7.6)$$

where $(\varepsilon_t)_{z=\frac{h}{2}}$ is the total relative elastic deformation of the end fibers

$$(\varepsilon_t)_{z=\frac{h}{2}} = \frac{M_x}{E_t W}.$$

Putting the value of E_t and $(\varepsilon_t)_{z=\frac{h}{2}}$ in the expression (7.6), we get

$$(\varepsilon_t)_z = \frac{M_x z}{I E_{prol}}\left(1 - k e^{-\alpha\left(\frac{t}{t_1}\right)^c}\right). \qquad (7.7)$$

Using the approximate value of the curvature $\frac{1}{\rho} = \frac{d^2y}{dx^2}$ and integrating the equation (7.5), the value of the total sag in any section of the bar and of the angle of deflection at any time interval t can be determined. For example, the value of the total sag in any section of the bar at any time interval may be obtained by using the graphical-analytical method

$$(f_t)_x = \frac{R_f x - F_x x_c}{E_t I}, \qquad (7.8)$$

where R_f is the fictituous supporting reaction; x is the distance of the fictituous reaction from the section; F_x is the area of diagram of bending moment up to the given section, due to the fictituous loading; and x_c is the distance from the center of gravity of the plane F_x to the given section.

The angle of deflection of the cross section, situated at a distance x from the origin of the coordinates, at the end of any time interval can be determined from the formula

$$(\theta_t)_x = \frac{R_f - F_x}{I E_t} \qquad (7.9)$$

Thus, in the case of bending of a bar under the action of a steady load, if the maximum stresses in the fibers do not exceed the limit of long-term resistance to compression, then sags and deflection angles of the sections change according to an exponential rule in the course of an infinitely long time.

On completely unloading, instantaneous redeformation, equal in magnitude to the elastic deformation obtained on loading, is formed in the beginning and then the process of redeformation, developing with time and ending in the course of the time interval of redeformation, equal to that of deformation, takes place.

EXPERIMENTAL INVESTIGATION ON CREEP DUE TO BENDING

In order to verify the laws of deformation and redeformation under a sustained acting load and the correctness of the law of proportionality between loads and sags with the passage of time, experimental investigations on creep due to bending were conducted in the elastic region of work of material. For this purpose, experiments were conducted on wood and laminated plastics.

The experimental curves of deformation and redeformation due to bending are shown in Fig. 37 as an example. The slight apparent residual deformation after unloading and holding is due to the weight of the measuring device and the gravitational weight of the objects under test which have not been taken into account and which, in fact, are not removed on unloading the sustained acting load.

From the results of the tests on bending, the value of instantaneous and prolonged moduli of elasticity were also determined. Some of their values and a comparison between instantaneous and prolonged moduli of elasticity in the case of compression and tension are given in Table 11. As seen, the values of the moduli of elasticity for the cases of compression, tension and bending are almost equal.

The proportionality law with the passage of time has been verified by comparison of the change with time of the ratio of elasto-viscous and elastic instantaneous sags for different time intervals of development of sag, under the action of different loads which do not give rise to residual plastic deformations.

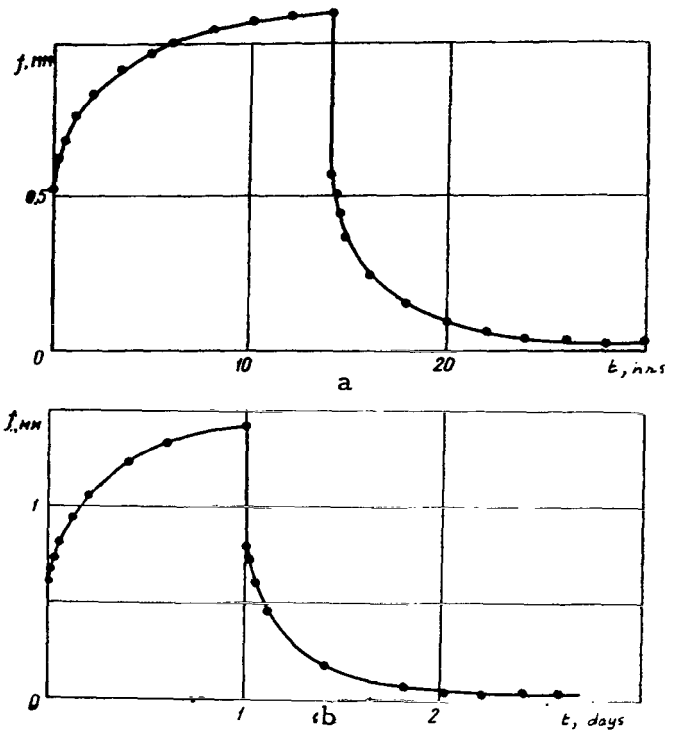


Fig. 37. Diagrams of deformations and redeformations in the elastic region of work of material on bending.

a — DSP-B parallel to the ply layers (M = 375 dkn. cm); b — KAST-V along warp (M = 230 dkn. cm).

Table 11

Material	Instantaneous modulus of elasticity $E_{inst} \times 10^{-5}$, dkn/cm^2			Prolonged modulus of elasticity $E_{prol} \times 10^{-5}$, dkn/cm^2		
	Bending	Compression	Tension	Bending	Compression	Tension
Pine wood	1.64	1.63	1.76	0.97	0.78	0.80
DSP-B	3.50	3.40	3.60	2.70	2.80	2.75
DAST-V	2.82	2.74	2.95	1.60	1.59	1.80

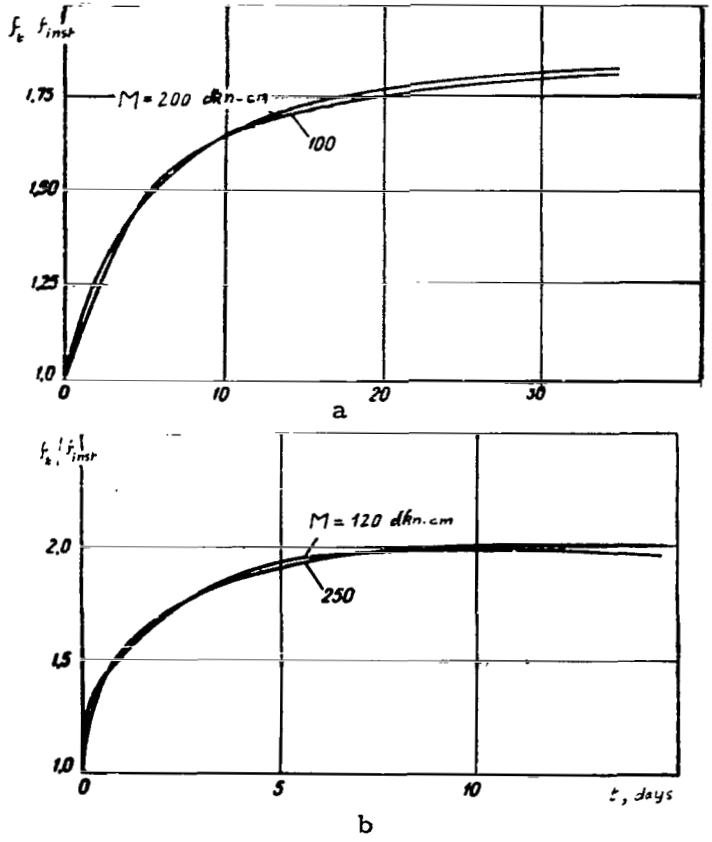


Fig. 38. Diagrams of the development of the ratio of total deformations to elastic instantaneous deformations in the elastic region of work of material on bending.

a - KAST-V along warp; b - DSP-V parallel to the ply layers.

Fig. 38 shows the change, with time, in the ratio of elasto-viscous and elastic instantaneous sags. As seen, the interval between the curves for different loads is very small, which is a sufficient proof of the correctness of the proportionality law with time in the case of bending in the elastic region of work of material.

CHAPTER VIII

CREEP DUE TO BENDING IN THE CASE OF DEVELOPMENT OF PLASTIC DEFORMATIONS IN COMPRESSED ZONE OF SECTION

CHANGE IN THE DIAGRAM OF NORMAL STRESSES WITH TIME

On applying a constant sustained load giving rise to stresses $\sigma > \sigma_{l.l.n.compr}$ in the end fibers of the danger section, but not exceeding $\sigma_{l.l.n.tens}$ in the course of an infinitely long time, instantaneous elastic deformations arise in all the fibers of the bar along the height after which elasto-viscous deformations develop with time.

Development of elasto-viscous deformations in all the fibers of the bar takes place in the course of the limiting time interval for the maximum compressed end fiber. In the course of this time interval, the stressed state and the laws of development of fibrous and total deformations are the same as in the case when the stresses in the maximum stressed fiber are less than or equal to the limit of long-term resistance to compression ($\sigma \leq \sigma_{l.l.n.compr}$). In this case, deformation, as mentioned above, obeys the proportionality law and follows the hypothesis of planar sections, but stresses vary along the height of the section of the bar according to a linear law. The angle of deflection of the section and the sag of the bar are determined by the formulas given in Chapter VII.

After a lapse of the time interval equal to the limiting time interval $t_{s\ compr}$ for the maximum compressed fiber, the process of the formation of elastic deformations in this fiber comes to an end, and the formation of plastic deformations begins. The diagram of normal stresses and the diagram of relative fibrous deformations at time interval $t = t_{s\ compr}$ have the form shown in Fig. 39. At the time interval $t = t_{s\ compr}$, the process of the development of elastic deformations also ceases in the neighboring less intensely stressed fibers and the formation of plastic deformations begins. The closer the compressed fiber to the neutral axis, the lesser are the stresses acting in it and the higher is the limiting time interval for it.

Nonvariability, with time, of the constant external bending load is related to the constancy, with time, of the value of

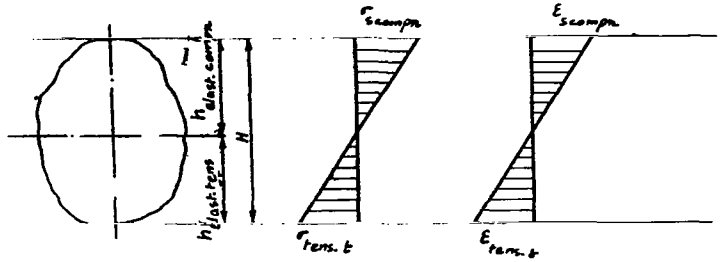


Fig. 39. Diagrams of normal stresses and relative fibrous deformation at time interval $t = t_{s\,compr}$.

the bending moment, acting in one or the other section of the bar. According to the condition of equilibrium of the external and internal forces at any time, the moment of the internal elastic forces should be equal to the bending moment in every section along the length of the bar. If the moment of internal elastic forces does not vary with time in any arbitrary section of the bar, the process of redistribution of internal elastic forces along the section begins with the formation of plastic deformations in the compressed zone of the bar. Consequently, the value of the internal elastic forces, which are the cause of plastic deformations in the compressed zone of the bar, and the value of internal elastic forces in the elastic core of the section, change. An increase in the height of the plastic core of the section of the bar and a corresponding decrease in the height of the elastic core take place simultaneously with the process of redistribution and change in the internal elastic forces.

At any time interval after the beginning of the formation of plastic deformations, a continuous process of redistribution of stresses occurs, during which the condition of equilibrium of the changing internal elastic forces is observed at every moment of time.

The process of redistribution of internal stresses continues for an infinitely long time, if the stresses in the stretched end fibers, increasing with time at the cost of a decrease in stresses in the plastic core of the compressed zone, do not exceed the limit of long-term resistance to tension. In this case, a change in the values of fibrous and total deformations with time also continues for an infinitely long time, but without leading to the fracture of the bar, although the formation of plastic deformations occur in the compressed zone of the bar. This is due to the fact that, during the change in the value of stresses,

the stresses in the stretched end fiber do not exceed $\sigma_{T.A.R.Compr}$ and consequently, the stretched end fiber does not fracture. Since the stretched fiber does not fracture in the course of infinitely long time of development of deformations, distribution of plastic deformations along the height of the section in the compressed zone of the bar, change in fibrous and total deformations, decrease in stresses in the compressed and increase in the stretched zones of the bar take place, with time, according to an asymptotic law, approaching the ultimate values in the course of an infinitely long time.

Since total deformations of the bar change in the course of an infinitely long time according to an asymptotic law, the plastic deformations in the compressed zone of the bar also change with time according to an asymptotic law. The asymptotic growth of plastic deformations under the action of a constant bending moment shows that the rate of growth also changes, with time, according to an asymptotic law from v_s at $t_{s\ compr}$ to $v_s \longrightarrow 0$ at $t \longrightarrow \infty$. The change in the rate of growth of plastic deformations with time leads to a change in stresses, under the action of which they develop. Consequently, these stresses also change continuously with time according to an asymptotic law from σ_s at v_s and t_s compr to $\sigma_s \rightarrow \sigma_{l.n.compr}$ at $v_s \longrightarrow 0$ and $t \longrightarrow \infty$.

Taking the hypothesis of planar section to be correct for elasto-plastic work of material of the bar as well, it may be noted that the free development of plastic deformations in the compressed zone is constrained and the plastic deformations themselves develop with time at a rate less than the rate of their development in the case of free deformation, unrelated to the hypothesis of planar sections.

Since total deformations of the bar change with time according to an asymptotic law and do not lead to fracture during an infinitely long time of their development, the angle of deflection of the section also changes according to an asymptotic law, when the hypothesis of planar sections is assumed. The angular rate of deflection of the section changes from $\omega \rightarrow \infty$ at $t \longrightarrow 0$ to $\omega \longrightarrow 0$ at $t \rightarrow \infty$. Since, for each time interval t, the angular deflection rate of the section has a definite value, the condition of proportionality for the values of the rate of development of deformation of the fibers at every time interval is more or less imposed by the hypothesis of planar sections.

Thus, at time interval t, which is higher than $t_{s\ compr}$ the limiting time interval of development of elastic deformations in the compressed end fiber — assuming that the hypothesis of planar sections holds good, the diagram of relative deformations of fibers and the corresponding diagram of the rate of development of deformation have the form shown in Fig. 40. In the case of fibers with the limiting elastic deformation ε_s increasing with time during the time interval $t > t_{s\ compr}$ at a rate v_s, the stresses are equal to the elastic limit at the end of this time interval. Stresses in the fibers, distributed within the limits of elastic cores of the stretched and compressed zones of section, are proportional to the values of relative deformations and change along the height of the section within the limits of the elastic core according to the linear law.

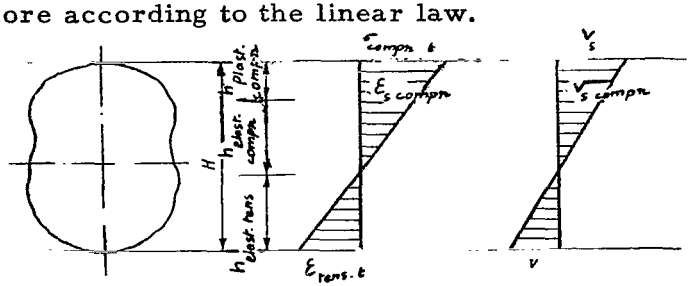


Fig. 40. Diagram of relative fibrous deformations and of the rates of their development if time interval $t > t_{s\,compr}$.

It is known that there is a functional relation between the rate of development of plastic deformations and the magnitude of stresses which cause these deformations and the relation differs from the proportionality law between the rate of development of elastic deformations at a given time interval and the stresses. Fig. 41 shows the interdependence of the elastic limit and the limiting rate of development of elastic deformations (rate of development of plastic deformations); the straight lines representing the dependence of the rate of development of elastic deformations on the stresses for any time interval of development of elastic deformations have also been drawn.

The point of intersection of the straight line with the curve gives the value of the limiting rate of development of elastic deformation for a fiber placed at the boundary of elastic and plastic core of the section, and the value of the stresses at the given time interval $t > t_{s\ compr}$.

The values of the rate of development of plastic deformations of fibers distributed within the limits of the plastic core of

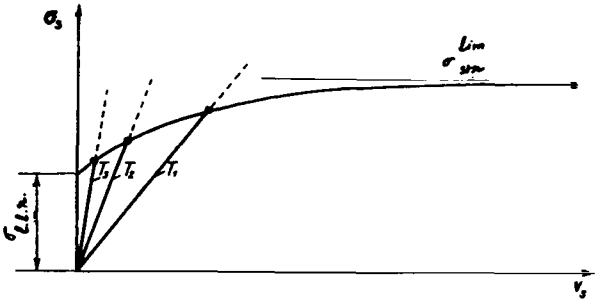


Fig. 41. Relation between elastic limit and the rate of development of plastic deformations.

the section are determined by the hypothesis of planar sections and by the condition of proportionality between the rate of their development and distance of the fibers from the neutral axis on the one hand, and by the relation between elastic limit and rate of development of plastic deformations on the other hand. Since at a given time interval $t > t_{s\ compr}$, the development of plastic deformations with rates proportional to the distance of the fibers from the neutral axis is possible at stresses lower than those determined from the proportionality law between σ and v, the stress diagram in the plastic core of the section deviates from the linear one. Consequently, if the hypothesis of planar sections is retained, normal stresses within the limits of the plastic core of the section change along the height of the plastic core according to the curvilinear law, determined by the relation between the stresses σ_s' and the rate of plastic deformation v_s'. For any fiber situated within the limits of the plastic core, the relation between stresses σ_s' and rate of development v_s of the limiting elastic deformations ε_s at the boundary between the elastic and the plastic core of the section may be established on the basis of the condition of proportionality between the rate of development of deformations and the distance from the neutral axis.

On the basis of what has been stated above, the diagram of normal stresses at the end of any time interval $t > t_{s\ compr}$ will assume the form shown in Fig. 42. It should be mentioned that the curvilinear portion of the diagram of normal stresses within the limits of the plastic core of the section differs only slightly from the vertical straight line, since its height and the difference between the stresses σ_s' and σ_s are insignificant.

By changing the time of action of the constant bending moment from $t = t_s$ to $t \to \infty$, the curvilinear part of the dia-

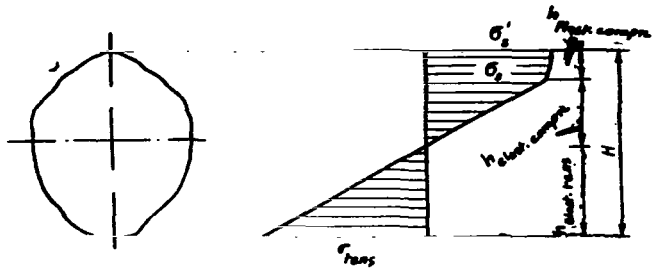


Fig. 42.Diagram of normal stresses at time of action of the load $t > t_{s\,compr}$

gram of normal stresses within the limits of the plastic core of the compressed zone changes with time, approaching the vertical rectilinear part at $t \to \infty$ at the rate of development of plastic deformations v = 0, which corresponds to the stresses $\sigma_{s\,compr} = \sigma_{l.l.n.\,compr}$ (Fig. 43).

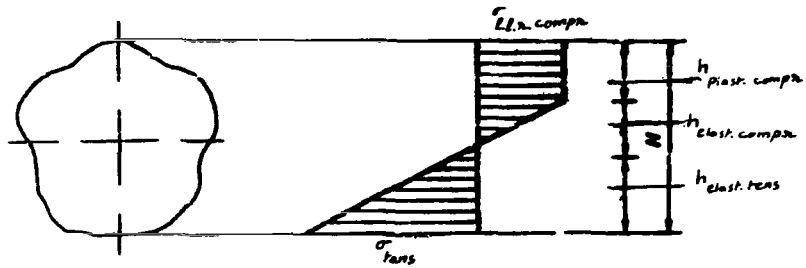


Fig. 43.Diagram of normal stresses at time of action of the load $t \to \infty$.

Thus, the portion of the diagram of normal stresses, changing with time, in the plastic core of the compressed zone of the section is expressed by curves changing with time, which successively approach the vertical straight line.

At any time interval $t_s < t < \infty$, the determination of the heights of plastic and elastic cores of the section and of the values σ_{tens} and σ_s' at a known constant bending moment and known values of elastic limit σ_s' for the given time interval, of limiting elastic deformation ε_s and their corresponding limiting rate v_s requires the solution of six equations:

1) sum of the projections of normal stresses on the x-axis is zero:

$$\sum x = 0; \tag{8.1}$$

2) sum of the moments of internal elastic forces is equal to the given constant external moment:

$$\sum M = M_{int} = const; \tag{8.2}$$

3) the height of the section is equal to the sum of the heights of plastic and elastic cores of the section:

$$H = h_{plast.\,compr} + h_{elast.\,compr} + h_{elast.\,tens}\ ; \tag{8.3}$$

4) according to Fig. 42:

$$\frac{\sigma_{s\,compr}}{\sigma_{tens}} = \frac{h_{elast.\,compr}}{h_{elast.\,tens}}\ ; \tag{8.4}$$

5) according to the rule of proportionality between the rates of development of deformations of fibers and the distance from the neutral axis (see Fig. 40):

$$\frac{v_s}{v_s'} = \frac{h_{elast.\,compr}}{-h_{plast.\,compr} + h_{elast.\,compr}}\ ; \tag{8.5}$$

6) relation between stresses σ_s' in the compressed end fiber and the rate v_s:

$$\sigma_s' = f\left(v_s\ ;\ \frac{h_{elast.\,compr}}{h_{plast.\,compr} + h_{elast.\,compr}}\right). \tag{8.6}$$

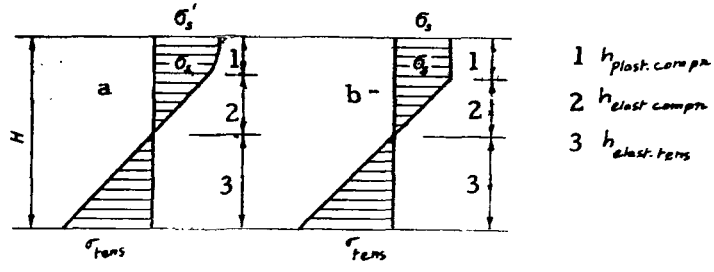


Fig. 44. Real (a) and simplified (b) diagrams of normal stresses.

By solving all the six equations simultaneously, the dimensions of elastic and plastic cores of the section can be determined and the diagram of distribution of normal stresses along the section can be established. For practical purposes, it is convenient to use simpler diagrams of normal stresses. Replacing the diagram (Fig. 44, a) by its equivalent trigonal-trapezoidal diagram (Fig. 44, b), as is usually accepted in theory of bending, a simpler diagram is obtained with the consideration of plastic deformations. The trigonal-trapezoidal diagram of the distribution of normal stresses along the section gives slight deviations in the dimensions of plastic and elastic core of the section and in the magnitude of stresses in the stretched end

fiber. However, the determination of these quantities in the case of trigonal-trapezoidal projection diagram of normal stresses requires solution of four equations only. Besides, such a substitution is possible also because the abovementioned (see Fig.42) diagram of normal stresses, continuously changing with time, approaches the trigonal-trapezoidal form at $t \to \infty$. However, it must be noted that the determination of deformations by using trigonal-trapezoidal projection diagrams involves a certain overstatement of values when the time interval $t_s < t < \infty$. At $t = t_s$ and $t \to \infty$, the values of deformations, determined by using both the diagrams, coincide. Within the limits $t = t_s$ and $t \to \infty$, deformations increase monotonically, and hence, there will not be sharp deviations in their values.

Thus, accepting the simplified trigonal-trapezoidal diagram of distribution of normal stresses, along the height of the section, we consider that stresses in the compressed end fiber and all other fibers within the limits of the plastic core of the section can be determined by the formula:

$$\sigma_{s\,compr} = \frac{\sigma_{l.l.n.\,compr}}{1-(1-k_2)\,e^{-\alpha\left(\frac{t}{t_1}\right)^c}} . \tag{8.7}$$

Dimensions of elastic and plastic cores of the section and the value of stresses in the stretched end fiber (see Fig. 44,b) may be obtained by solving the equations (8.1) to (8.4) simultaneously.

Let us denote the ratio of the stresses in the compressed end fiber to the stresses in the stretched end fiber at time t by

$$\eta_t = \frac{\sigma_{t\,tens}}{\sigma_{s\,compr}} = \frac{h_{elast.\,tens}}{h_{elast.\,compr}} . \tag{8.8}$$

The stresses within the limits of elastic core of the section as a function of y, the distance from the stretched end fiber, are

$$\sigma = \sigma_{s\,compr}\left(\eta_t - \frac{y}{h_{elast.\,compr}}\right). \tag{8.9}$$

Proceeding from the diagram of normal stresses (see Fig. 44, b) it is possible to write down two equations:

$$\left.\begin{aligned} H &= h_{Plast.\,compr} + h_{elast.\,compr} + h_{elast.\,tens}\,, \\ h_{elast.\,tens} &= \eta_t\, h_{elast\,compr}\,. \end{aligned}\right\} \tag{8.10}$$

Let us introduce the following symbols: F - area of cross section; F_1 - area of the elastic core of the section; S_{1x} and I_{1x} - static moment and moment of inertia of the elastic core about x-axis; W - moment of resistance of the section; S_x - static moment of the section about x axis.

The equation of projections along x-axis is

$$F - F_1(\eta_t + 1) + \frac{S_{1x}}{h_{elast.\ compr}} = 0 \qquad (8.11)$$

From the equation (8.11), the height of the compressed part of the elastic core is obtained as

$$h_{elast.\ compr} = \frac{S_{1x}}{F_1(\eta_t + 1) - F} \qquad (8.12)$$

Using the equation (8.10), we get the height of the stretched part of the elastic core as

$$h_{elast.\ tens} = \frac{S_{1x}\eta_t}{F_1(\eta_t + 1) - F} \qquad (8.13)$$

and the height of the plastic core

$$h_{plast.\ compr} = H - \frac{S_{1x}(\eta_t + 1)}{F_1(\eta_t + 1) - F} \qquad (8.14)$$

According to the theory of bending with due consideration of the plastic deformations in the compressed zone of the bar, the constant moment of internal forces in the given section of the bar is

$$M_x = \mu_t \sigma_{s\,compr} W_{compr}, \qquad (8.15)$$

where μ_t is the coefficient, which takes into account the increase in the carrying capacity in the case of bending -- if plastic deformations have developed in a part of the section -- over the limiting elastic carrying capacity of the bar.

Bending moment does not change with time and hence the value of the coefficient μ_t must change with time, because the quantity η_t and the dimensions of the plastic core in the compressed zone and the dimensions of elastic cores in the compressed and stretched zones of the beam change with time.

The formula for obtaining the coefficient μ_t, changing with time, has the form

$$\mu_t = \frac{I_{1x}[F_1(\eta_t+1)-F]}{S_{1x}W_{compr}} - \frac{S_{1x}(\eta_t+1)-S_x}{W_{compr}}. \qquad (8.16)$$

Putting the values of η_t and μ_t in formula (8.15) and solving with respect to stresses in the stretched end fiber, their value can be obtained as a function of the bending moment and the stresses in the plastic core of the compressed zone

$$\sigma_{t\,tens} = f(M_x, \sigma_{s\,compr}) \qquad (8.17)$$

and the expression for the determination of the coefficient η_t as a function of the bending moment M_x and the stresses $\sigma_{s\,compr}$ may be obtained:

$$\eta_t = f(M_x, \sigma_{s\,compr}). \qquad (8.18)$$

The expression (8.18) is essential for determining the bending deformations in the elasto-visco-plastic region of work of the material.

The dimensions of the elastic and plastic cores of the section and their geometrical properties are expressed by the ratio η_t and the dimensions of the section. The dimensions of the section in (8.16) are reduced by putting the geometrical properties, and the coefficient μ_t is expressed only as a function of η_t for each particular case (as will be seen later). Thus, the formula (8.16) makes the determination of the value of the coefficient μ_t theoretically possible for any type of section, having an axis of symmetry lying on the plane of action of the forces. If the ratio η_t can change theoretically within wide limits from 1 to ∞, the coefficient μ_t changes within small limits according to a hyperbolic law approaching a value which is constant for each type of section

$$\mu_{lim} = \frac{S_x}{W_{compr}} \qquad (8.19)$$

Here S_x is the static moment of the section about the axis, tangent to the stretched end fiber, and W_{compr} is the moment of resistance of the section about the compressed end fiber.

The value of μ_{lim} corresponds to the case, when normal stresses, equal to elastic limit for compression, act throughout the area of the section of the bar. It is understandable that such a case does not exist for bars of an elasto-visco-plastic material.

VARIATION, WITH TIME, IN THE PROJECTION DIAGRAM OF NORMAL STRESSES FOR BARS OF A RECTANGULAR CROSS SECTION

Let us see how the diagram of normal stresses changes for bars with a rectangular cross section, if plastic deformations develop only in the fibers of the compressed zone of the section and the stretched fibers act elastically.

Using the formula given above, putting in them the geometrical properties of rectangular cross section and assuming that the coefficient η_t changes with time, we find that the value of the coefficient μ_t at time t is

$$\mu_t = \frac{3\eta_t - 1}{\eta_t + 1}, \tag{8.20}$$

and the moment of the internal elastic forces is equal to the moment of the external forces:

$$M_x = \mu_t \sigma_{s\,compr} W. \tag{8.21}$$

Putting the values of η_t and μ_t from the formulas (8.8) and (8.20) in formula (8.21) and solving with respect to stresses in the stretched end fiber, we get

$$\sigma_{t\,tens} = \frac{M_x \sigma_{s\,compr} + (\sigma_{s\,compr})^2 W}{3\sigma_{s\,compr} W - M_x} \tag{8.22}$$

or, by substituting the value of $\sigma_{s\,compr}$ according to the formula (8.7), we get

$$\sigma_{t\,tens} = \frac{M_x \left[1-(1-k_2)\,e^{-\alpha\left(\frac{t}{t_1}\right)^c}\right] + \sigma_{l.l.n.\,compr} W}{3\sigma_{l.l.n.\,compr} W - M_x \left[1-(1-k_2)\,e^{-\alpha\left(\frac{t}{t_1}\right)^c}\right]} \times$$

$$\times \frac{\sigma_{l.l.n.\,compr}}{\left[1-(1-k_2)\,e^{-\alpha\left(\frac{t}{t_1}\right)^c}\right]} \tag{8.23}$$

The ratio η_t after putting the value $\sigma_{s\,tens}$ is

$$\eta_t = \frac{M_x + \sigma_{s\,compr} W}{3\sigma_{s\,compr} W - M_x} =$$

$$= \frac{M_x \left[1-(1-k_2)\, e^{-\alpha\left(\frac{t}{t_1}\right)^c}\right] + \sigma_{\ell.\ell.n.\,compr} W}{3\,\sigma_{\ell.\ell.n.\,compr} W - M_x \left[1-(1-k_2)\, e^{-\alpha\left(\frac{t}{t_1}\right)^c}\right]} \qquad (8.24)$$

The dimensions of the height of the plastic core, heights of the compressed and the stretched parts of the elastic core and displacement of the neutral axis n at time t are determined according to formulas, obtained by the simultaneous solution of the equations (8.1) to (8.4) under the condition (8.8):

$$h_{plast.\,compr} = h\,\frac{\eta_t - 1}{\eta_t + 1} \;;\; h_{elast.\,compr} = \frac{2h}{(\eta_t+1)^2} \;;$$

$$h_{elast.\,tens} = \frac{2h\eta_t}{(\eta_t+1)^2} \;;\quad u = \frac{h(\eta_t-1)^2}{2(\eta_t+1)^2} \;. \qquad (8.25)$$

The value of η_t from formula (8.24) should be put in the formula (8.25) in order to obtain the geometrical properties of the section.

Since the process of formation and development of deformations in the case under consideration takes place for an infinitely long time, the stresses in the compressed end fiber at $t \to$ will be

$$\sigma_{s\,compr} \longrightarrow \sigma_{\ell.\ell.n.\,compr} \qquad (8.26)$$

stresses in the stretched end fiber will be

$$\sigma_{t\,tens} \longrightarrow \frac{M_x + \sigma_{\ell.\ell.n.\,compr} W}{3\sigma_{\ell.\ell.n.\,compr} - M_x}\,\sigma_{\ell.\ell.n.\,compr} \qquad (8.27)$$

The ratio η_t at $t \to \infty$ is

$$\eta_t \longrightarrow \frac{M_x + \sigma_{\ell.\ell.n.\,compr} W}{3\sigma_{\ell.\ell.n.\,compr} W - M_x} \;. \qquad (8.28)$$

The height of the plastic core of the section at $t \to \infty$ is

$$h_{plast.\,compr} \to h\,\frac{M_x - \sigma_{l.l.r.\,compr} W}{2\sigma_{l.l.r.\,compr} W}. \qquad (8.29)$$

The height of the compressed part of the elastic core of the section at $t \to \infty$ is

$$h_{elast.\,compr} \to 2h\left(\frac{3\sigma_{l.l.r.\,compr} W - M_x}{4\sigma_{l.l.r.\,compr} W}\right)^2 = \frac{h}{8}\left(\frac{3\sigma_{l.l.r.\,compr} W - M_x}{\sigma_{l.l.r.\,compr} W}\right)^2. \qquad (8.30)$$

The height of the stretched part of the elastic core of the section at $t \to \infty$ is

$$h_{elast.\,tens} \to \frac{h}{8}\,\frac{(3\sigma_{l.l.r.\,compr} W - M_x)(M_x + \sigma_{l.l.r.\,compr} W)}{\sigma_{l.l.r.\,compr} W}. \qquad (8.31)$$

The displacement of the neutral axis at $t \to \infty$ is

$$u \to \frac{h}{8}\,\frac{(M_x - \sigma_{l.l.r.\,compr} W)^2}{(\sigma_{l.l.r.\,compr} W)^2}. \qquad (8.32)$$

DIFFERENTIAL EQUATION OF THE BENT AXIS OF THE BAR

In the case of a constant sustained load, after a lapse of the time interval $t_{s\,compr}$, plastic deformations begin to develop in the section, for which the bending moment for the given scheme of the bar and the load does not have the highest value. Taking the hypothesis of planar sections to be correct, we find that towards the end of any time interval up to $t = t_{s\,compr}$, the differential equation of the bent axis of the beam has the form

$$E_t I \frac{d^2y}{dx^2} = M_x, \qquad (8.33)$$

and the angles of deflection of the section and sag in any section are obtained by integrating this equation.

The modulus of elasticity E_t which changes with time, attains the limiting value E_s at the time interval $t = t_{s\,compr}$. At the time interval $t = t_{s\,compr}$ the differential equation of the bent axis has the form:

$$E_s I \frac{d^2y}{dx^2} = M_x. \qquad (8.34)$$

The appearance of plastic deformations in the compressed zone and their propagation along the height of the section and the length of the bar leads to the result that the proportionality law between stresses and deformations at any time interval does not hold good, and consequently, the differential equation (8.33) cannot be used to determine the sags and the angles of deflection.

Taking the form of the diagram of distribution of normal stresses due to bending in the elasto-visco-plastic region of work of material as trigonal-trapezoidal, and assuming that the region of propagation of plastic deformations, within the limits of which the diagram is accepted, lies along the length of the bar, and also assuming that the hypothesis of planar sections is practically valid, the expression for curvature may be written as

$$\frac{1}{\rho_t} = \frac{\varepsilon_{compr\ t} + \varepsilon_{tens\ t}}{h}, \qquad (8.35)$$

where $\varepsilon_{compr\ t}$ is the relative deformation of the compressed fiber at $t > t_{s\ compr}$ and $\varepsilon_{tens\ t}$ is the relative deformation of the stretched end fiber at $t > t_{s\ compr}$.

According to Fig. 40

$$\frac{\varepsilon_{compr\ t}}{\varepsilon_{tens\ t}} = \frac{h_{plast.\ compr} + h_{elast.\ compr}}{h_{elast.\ tens}}, \qquad (8.36)$$

from which

$$\varepsilon_{compr\ t} = \frac{h_{plast.\ compr} + h_{elast.\ compr}}{h_{elast.\ tens}} \varepsilon_{tens\ t}. \qquad (8.37)$$

Relative deformation of the stretched end fiber is

$$\varepsilon_{tens\ t} = \frac{\sigma_{tens\ t}}{E_t} \qquad (8.38)$$

Using the formulas (8.35) and (8.38), we get that the curvature

$$\frac{1}{\rho_t} = \frac{\sigma_{tens\ t}}{h_{elast.\ tens} E_t}. \qquad (8.39)$$

Since

$$\frac{\sigma_{tens\ t}}{\sigma_{s\ compr}} = \eta_t \quad and \quad \frac{h_{elast.tens}}{h_{elast.compr}} = \eta_t ,$$

the curvature is

$$\frac{1}{\rho_t} = \frac{\sigma_{s\,compr}}{h_{elast.\,compr} E_t} . \tag{8.40}$$

As fibrous deformations are small, the curvature $\frac{1}{\rho_t} = y''$ approximately, and the differential equation of bending obtained by taking into consideration the plastic deformation into the compressed zone of the section will be

$$y'' = \frac{\sigma_{s\,compr}}{h_{elast.\,compr} E_t} . \tag{8.41}$$

Expressing the dimensions of the compressed part of the core $h_{elast.\ compr}$ as a function of the bending moment $M_{(x)}$ and applying the usual methods of integrating differential equation, we obtain the angles of deflection and sag and their change with time in the case of development of plastic deformations in the compressed zone due to bending.

ANALYTICAL AND GRAPHICAL-ANALYTICAL METHODS FOR DETERMINING DEFORMATIONS OF BARS OF A RECTANGULAR SECTION

Let us consider the determination of deformations produced by a constant external load, acting in the course of time $t > t_{s\,compr}$.

Since the bending moment $M_{(x)}$ is variable along the length of the bar (except in the case of pure bending), the value of η_t can be obtained as a function of $M_{(x)}$.

The bending moment according to the diagram of normal stresses in the section (x) for the time interval $t > t_{s.\,compr}$.

$$M_{(x,t)} = \sigma_{s\,compr} W \frac{3\eta_t + 1}{\eta_t + 1} . \tag{8.42}$$

Assuming that the limiting elastic moment is

$$M_{elast} = \sigma_{s\,compr} W ,$$

we get

$$\eta_t = \frac{\frac{M_{(x,t)}}{M_{elast}} + 1}{3 - \frac{M_{(x,t)}}{M_{elast}}}. \qquad (8.43)$$

The dimension of the compressed part of the elastic core is

$$h_{elast.\,compr} = \frac{2h}{(\eta_t + 1)^2} \qquad (8.44)$$

Putting the value of $h_{elast.\,compr}$ in the formula (8.41), we get the differential equation of the bent axis

$$y'' = \frac{4M_{elast}}{E_t I\left(3 - \frac{M_{(x,t)}}{M_{elast}}\right)^2}. \qquad (8.45)$$

The equation (8.45) is true for the portion of the bar in which plastic deformations in the compressed zone have developed under the action of a constant load. For the portions of the bar, in which the acting bending moment is less than the limiting moment for the given time interval of the elastic bending moment, the differential equation

$$y'' = \frac{M_{(x,t)}}{E_t I}.$$

should be used.

Determination of sags and angles of deflection in the entire length of the bar requires both the equations to be integrated and consequently the analytical method of determination of deformations requires the determination of the constants of integration C_1 and C_2 for the equation (8.33) and C_3 and C_4 for the equation (8.45).

In order to determine the constants of integration, besides the usual conditions, we introduce additional conditions based on the fact that at the boundary of elastic and elasto-plastic regions of the bar, sags and angles of deflection determined by both the equations are equal ($x = x_a$; $y'_{elast} = y'_{elast.\,plast}$; $y_{elast} = y_{elast\ plast}$). The coordinate x_a is the distance from the origin of coordinates to the end of the elastically working portion of the bar.

Let us determine the constants C_1, C_2, C_3, C_4 by taking an example of a bar of a rectangular cross section resting on two supports and subjected to bending by a concentrated force at the middle of the span (Fig. 45).

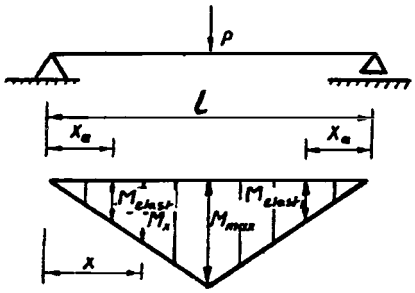


Fig. 45. Layout of the bar considered by way of an example for determining deformations by the analytical method.

The bending moment at the section x is

$$M_{(x)} = \frac{Px}{2}$$

The bending moment at the middle of the span is

$$M_{max} = \frac{Pl}{4}.$$

Let us denote the ratio by

$$\frac{M_{max}}{M_{elast}} = \mu_t \tag{8.46}$$

$$\frac{M_{(x)}}{M_{elast}} = \mu_t \frac{2x}{l}. \tag{8.47}$$

The differential equation of the bent axis for the elasto-plastic region of material of the bar is

$$y'' = \frac{Pl}{E_t I \mu_t \left(3 - \frac{2x}{l}\mu_t\right)^2}. \tag{8.48}$$

By integrating, we get

$$y' = \frac{Pl^2}{2E_t I \mu_t^2 \left(3 - \frac{2x}{l}\mu_t\right)} + C_1, \tag{8.49}$$

$$y = -\frac{P\ell^3}{4E_t I\mu_t^3}\ln\left(3 - \frac{2x}{\ell}\mu_t\right) + C_1 x + C_2 . \qquad (8.50)$$

The differential equation for the parts of the bar, working elastically, is

$$y'' = \frac{M_{(x)}}{E_t I} = \frac{Px}{2E_t I} . \qquad (8.51)$$

Integrating, we get

$$y' = \frac{Px^2}{4E_t I} + C_3 , \qquad (8.52)$$

$$y = \frac{Px^3}{12E_t I} + C_2 x + C_4 . \qquad (8.53)$$

The conditions for the determination of the integration constants are

$$\left.\begin{array}{ll} \text{when } x=0, & y_{elast} = 0; \\ \text{when } x = \frac{\ell}{2}, & y'_{elast.plast} = 0; \\ \text{when } x = x_a, & y'_{elast} = y'_{elast.plast}; \\ \text{when } x = x_a, & y_{elast} = y_{elast.plast} . \end{array}\right\} \qquad (8.54)$$

The distance x_a may be found out from the fact that in the elastic work of material of the bar

$$M_{elast} = \frac{Px_a}{2} ,$$

i.e.

$$x_a = \frac{2M_{elast}}{P} = \frac{\ell}{2\mu_t} .$$

After determining the constants of integration and putting their values in the equations (8.49), (8.50),(8.52) and (8.53), we get the expressions for determining the angle of deflection of

the section and the sag in the section x within the limits of the portions of the bar, working elastically:

$$y'_{x\,elast} = \frac{Px^2}{4E_t I} + \frac{P\ell^2(1-3\mu_t)}{16E_t I\mu_t^2(3-\mu_t)}, \qquad (8.55)$$

$$y_{x\,elast} = \frac{Px^3}{12E_t I} + \frac{P\ell^2 x(1-3\mu_t)}{16E_t I\mu_t^2(3-\mu_t)}, \qquad (8.56)$$

and also within limits of the portions of the bar, working elasto-plastically:

$$y_{x\,elast.plast} = \frac{P\ell^2}{2E_t I\mu_t^2}\left[\frac{1}{3-\frac{2x}{\ell}\mu_t} - \frac{1}{3-\mu_t}\right], \qquad (8.57)$$

$$y_{x\,elast.plast} = \frac{P\ell^3}{48E_t I\mu_t^3}\left[5 - \frac{24\mu_t x}{\ell(3-\mu_t)} + 12\ln\frac{2}{3-\frac{2\mu_t x}{\ell}}\right]. \qquad (8.58)$$

The change in the angle of deflection and the sag, with time, produced by a constant external load may be determined by using the given formulas. For this purpose, it is essential to know the law of the change in the modulus of elasticity E_t and the coefficient μ_t, which depends on the change in elastic limit of compressed fibers with time.

In order to determine deformations in the case of elasto-plastic work of material of the bar, it is also possible to use graphical-analytical method.

Let us find the deformations of the bar of a rectnagular cross section.

As mentioned earlier, the value of the curvature in the section x of the elasto-plastically deformed bar at time $t > t_{s\,compr}$ may be determined by the formula (8.40).

Let us exchange the real bar, working in the elasto-plastic region, by a fictituous one, whose material works elastically throughout the entire section. In order to determine the fictituous bending moment for the elasto-plastically stressed state of the bar in the section x at time t, we substitute the trapezoid-

al stress diagram in the cross section by the fictitious stress diagram for the elastic stressed state of the fictitious bar in the same section x and at the same time t.

The graphical-analytical method of determination of deformations due to bending, which takes plastic deformations into consideration, is based on the assumption of unlimited elastic work of material of the fictitious bar. It is known that deformations due to bending, with the formation of plastic deformations increase faster than during the elastic work of the bar. Consequently, in order to obtain the real deformation beyond the limit of elastic work by considering the material as elastic, it is essential to introduce a reduced moment greater than the real moment of the acting load. The reduced moment gives rise to a deformation of the bar of elastic material, which is equal to the deformation of a bar of elasto-plastic material. Let us plot a reduced diagram with the reduced moments and consider the bar as elastic, but having a load which differs from the real load.

To obtain the value of the reduced moment, let us draw a line through the center of the cross section of the bar (Fig. 46) parallel to the elastic part of the trapezoidal stress diagram which depicts the linear law of distribution of fictitious stresses. On the basis of the proportionality law between stresses and elastic deformations with time, it may be considered that the cross section of real and fictitious bars at the given time t turns through the same angle (sections are parallel), and hence, their radii of curvature are the same in the given section and at the given time.

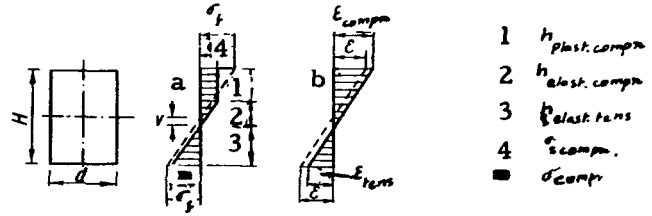


Fig. 46. Determination of deformations by graphical-analytical method.
a - diagram of normal stresses; b - diagram of relative elongation.

The curvature of the real beam in the section x at time t is determined from formula (8.40).

The curvature of the fictitious bar in the section x at time t is

$$\frac{1}{\rho_t} = \frac{(M_x)_f}{E_t I}. \tag{8.59}$$

Equating the curvature of real and fictitious bars, we get the expression for the determination of fictitious bending moment

$$(M_x)_f = \frac{\sigma_{scompr} I}{h_{plast.\,compr}}. \tag{8.60}$$

Substituting in the expression the equation (8.24) for $h_{elast.\,compr}$(8.25), we get

$$h_{elast.\,compr} = \frac{h(3\sigma_{scompr} W - M_x)^2}{8(\sigma_{scompr} W)^2} \tag{8.61}$$

Putting the expression (8.61) in the formula (8.60), we get the fictitious bending moment

$$(M_x)_f = \frac{4(\sigma_{scompr} W)^3}{(3\sigma_{scompr} W - M_x)^2} = \frac{4\sigma_{scompr} W}{\left(3 - \frac{M_x}{\sigma_{scompr} W}\right)^2}. \tag{8.62}$$

Change in the moment in the reduced moment diagram compared to that in the actual diagram in any section of the bar at any time $t > t_s$ is

$$\Delta M_x = (M_x)_f - M_x \tag{8.63}$$

or

$$\Delta M_x = \frac{4\sigma_{scompr} W}{\left(3 - \frac{M_x}{\sigma_{scompr} W}\right)^2} - M_x. \tag{8.64}$$

The distance between the boundary of elastic and elasoplastic parts of the bar and the support may be determined by equating the bending moment of the external load M_x to the fictitious bending moment in the same section ($M_x)_f$:

$$M_x = (M_x)_f \tag{8.65}$$

or

$$M_x = \frac{4\sigma_{scompr} W}{\left(3 - \frac{M_x}{\sigma_{scompr} W}\right)^2}. \tag{8.66}$$

Let us put the cubic equation (8.66) in the form

$$\left(\frac{M_x}{\sigma_{s\,compr}\,W}\right)^3 - 6\left(\frac{M_x}{\sigma_{s\,compr}\,W}\right)^2 + 9\,\frac{M_x}{\sigma_{s\,compr}\,W} - 4 = 0. \tag{8.67}$$

Solving the equation (8.67), we get

$$M_x^{I} = 4\,\sigma_{s\,compr}\,W \quad \text{and} \quad M_x^{II,III} = \sigma_{s\,compr}\,W.$$

Since, at the boundary of elastically and elasto-plastically deformed parts of the bar, the stresses in the end fiber cannot be four times higher than the elastic limit of the compressed fiber, we assume

$$M_x = \sigma_{s\,compr}\,W. \tag{8.68}$$

Putting the value of M_x expressed in terms of the external load and the distance of the section $x = x_a$ in the equation (8.68) and solving with respect to x_a, the distance from the support of the bar to the boundary between the elastic and elasto-plastic parts of the bar at time t can be obtained. Since the value of $\sigma_{s\,compr}$ changes with time, the distance x_a also changes with time at constant values of bending moments of the external forces.

If the bending moment diagram consists of a number of parts, then in order to determine the values of x_a, a straight line, parallel to the base of the diagram of real moments, is drawn on the diagram of moments at a distance $M_x = \sigma_{s\,compr} \cdot W$ from the direction of the compressed fibers. The distances of the points of intersection from the reference point of the length give the position of the boundary between elastically and elasto-plastically deformed parts of the bar.

Within the limits of the parts of the bar acting elastically, the bending moment diagram is drawn by ordinary methods used for an elastically working bar. The bending moment diagram, at the parts of the bar working elasto-plastically, may be plotted as reduced bending moment diagram according to the formula (8.66). In this part, the normal bending moment diagram of an elastic beam is transformed in such a way, that each of its ordinates is equal to the sum of the bending moment of the elastically working bar and the bending moment which takes the plastic work of material into account.

The angle of deflection of the cross section at a distance x at any time interval t is equal to the fictitious transverse force in the section x from a fictitious load:

$$(\theta_t)_x = \frac{Q_x^f}{IE_t} . \tag{8.69}$$

The sag of the bar at a distance x at time t is equal to the fictitious bending moment in the section x from a fictititous load:

$$(f_t)_x = \frac{M_x^f}{IE_t} , \tag{8.70}$$

where

$$M_x^f = R_f x - F_x x_c$$

In the formulas (8.69) and (8.70), the area of the moment diagram is

$$F_x = \int_0^{x_a} M_x dx + \int_{x_a}^{x} (M_x)_f dx . \tag{8.71}$$

The distance of the given section from center of gravity of the area F_x is

$$x_c = \frac{\int_0^{x_a} M_x x\, dx + \int_{x_a}^{x} (M_x)_f x\, dx}{F_x} . \tag{8.72}$$

Let us determine the deformation of the bar in the case of pure bending at any time $t > t_{s\,compr}$. In the case of pure bending (Fig. 47, a), the fictitious bending moment is

$$(M_x)_f = \frac{4\sigma_{s\,compr} W}{\left(3 - \frac{M}{\sigma_{s\,compr} W}\right)^2} , \tag{8.73}$$

where

$$M_x = M = const .$$

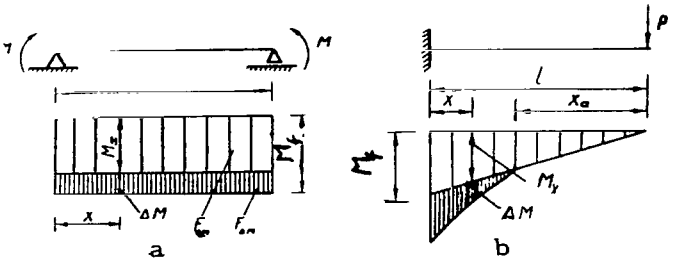


Fig. 47. Layout of bars considered by way of an example for determining deformations by graphical-analytical methods.
a - pure bending; b - bending of a cantilever.

Increase in the moment in the reduced moment diagram as compared to the real one changes with time due to formation of plastic deformations in the compressed zone of the bar:

$$\Delta M_t = (M_x)_f - M \cdot \tag{8.74}$$

At interval $t = t_{s\,compr}$, $\Delta M_t = 0$ and at time interval $t \longrightarrow \infty$, the value of the increase in moment $\Delta M_{t\to\infty}$ tends to maximum value:

$$\Delta M_{t\to\infty} = \frac{4\sigma_{l.l.r.compr} W}{\left(3 - \frac{M}{\sigma_{l.l.r.compr} W}\right)^2} - M. \tag{8.75}$$

The angle of deflection of any section at a distance x at any time interval $t = t_{s\,compr}$ is

$$(\theta_t)_x = \frac{4\sigma_{s\,compr} W\left(\frac{l}{2} - x\right)}{\left(3 - \frac{M}{\sigma_{s\,compr} W}\right)^2 E_t I} \tag{8.76}$$

If the time interval is changed from $t = t_{s\,compr}$ to $t \longrightarrow \infty$, the angle of deflection increases according to an exponential law.

The sag in any section at a distance x at any time $t > t_{s\,compr}$, according to formula (8.70), is

$$(f_{t\,s\,compr})_x = \frac{2\sigma_{s\,compr} W x(l-x)}{\left(3 - \frac{M}{\sigma_{s\,compr}}\right)^2 E_t I} \tag{8.77}$$

If the time interval is changed from $t = t_{s\ compr}$ to $t \to \infty$, the sag in the section x also increases according to an exponential law.

As an example, let us consider the determination of deformations at any time $t = t_{s\ compr}$ in the case of bending of a cantilever by a concentrated force, applied at the end of the cantilever (Fig. 47,b).

The bending moment of the load in any section is

$$M_x = -Px.$$

Let us find out the boundary between the elastically and elasto-plastically deformed parts of the bar. For this, a line is drawn at a distance $M = \sigma_{s\,compr} W$ parallel to the base of the bending moment diagram, which cuts the diagram into two parts. From similarlity of triangles, we get

$$x_a = \frac{\sigma_{s\,compr} W}{P}.$$

In the part of the bar from $x = x_a$ to $x = l$, the increase in the bending moment due to the formation of plastic deformations is added to the real moment from external forces M_x

$$(\Delta M_t)_x = (M_f)_t + Px, \qquad (8.78)$$

where

$$(M_f)_t = \frac{4\sigma_{s\,compr} W}{\left(3 - \frac{Px}{\sigma_{s\,compr} W}\right)^2}.$$

Since the stresses $\sigma_{s\ compr}$ change with time, the reduced moment in the section x changes with the change in time interval from $t = t_{s\,compr}$ to $t \to \infty$ according to an exponential law.

According to formula (8.69), the angle of deflection of any section at a distance $x > x_a$ is

$$(\theta_t)_x = -\frac{F_x}{E_t I}, \qquad (8.79)$$

where F_x, according to formula (8.71), is

$$-F_x = -\int_0^{x_a} px\,dx - \int_{x_a}^{x} \frac{4\sigma_{scompr}W}{\left(3-\frac{P_x}{\sigma_{scompr}W}\right)^2}\,dx.$$

Integrating, we get

$$(\theta_t)_x = -\frac{3(\sigma_{scompr}W)^2}{2PEI}\left[1-\frac{8\sigma_{scompr}W}{3(3\sigma_{scompr}W-Px}\right]. \quad (8.80)$$

According to formula (8.70), the sag in any section at a distance $x > x_a$ is

$$(f_t)_x = \frac{F_x x_c}{E_t I} \quad (8.81)$$

Let us find out the value

$$F_x x_c = \int_0^{x_a} px^2\,dx + \int_{x_a}^{x} \frac{4\sigma_{scompr}Wx}{\left(3-\frac{Px}{\sigma_{scompr}W}\right)}\,dx.$$

Integrating the expression for $F_x x_c$ and putting the result in the expression (8.81), we get

$$(f_t)_x = -\frac{4(\sigma_{scompr}W)^3}{p^2E_tI}\left(\frac{13}{12}-\frac{Px}{\sigma_{scompr}W}+3\ln\frac{2}{3-\frac{Px}{\sigma_{scompr}W}}\right) \quad (8.82)$$

Experimental verification of the laws of creep has been conducted in the case of pure bending of bars of pine wood and DSP-laminated plastic. Fig. 48 shows the diagram of growth, with time, of the ratio of total and instantaneous elastic deformations at different loads higher than the limiting elastic load. As seen, with the passage of time, the law of proportionality holds good for a certain part, after which deviation is observed, and the higher the load, the more is this deviation.

Thus, experimental verification confirmed the asymptotic law of development of deformations mentioned earlier as well

as the increase in deformation due to the appearance of plastic deformations in the compressed zone of the beams.

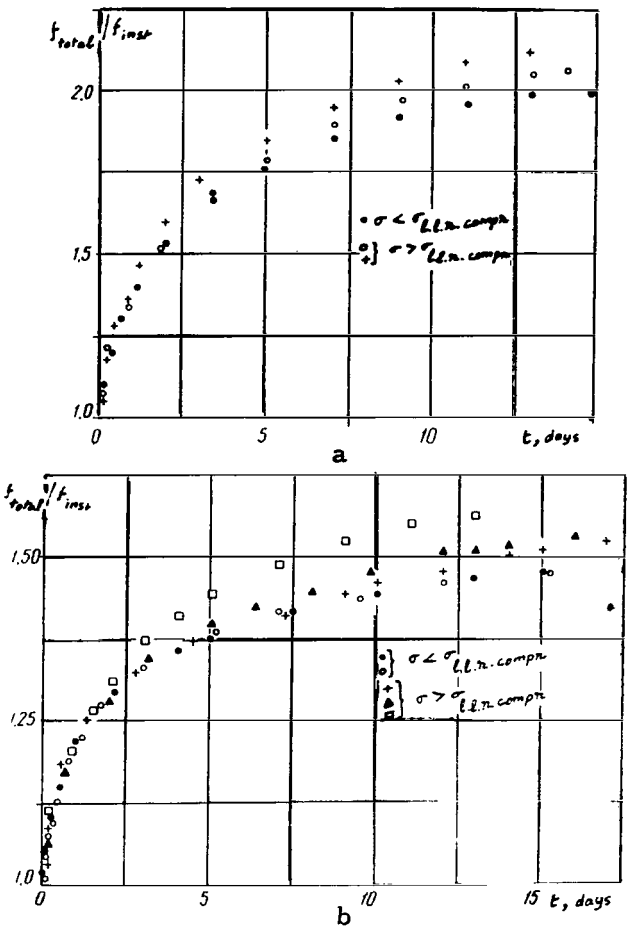


Fig. 48. Diagrams of growth, with time, of the ratio of the total deformations to the elastic instantaneous bending deformations of a) DSP-B plastic, and b) pine wood.

DETERMINATION OF DEFORMATIONS OF BARS OF AN ARBITRARY CROSS SECTION

In the case of bending of a bar of arbitrary section, the differential equation of the bent axis for the part, at which plastic deformation develops in the compressed zone, has the form:

$$y'' = \frac{\sigma_{scompr}}{h_{elast.\,compr} E_t} .$$

The reduced fictitious moment M_f, which is the basic characteristic of the graphical-analytical method for determining the deformation, can be determined, as in the case of rectilinear cross section, from the equality of real and fictitious curvature:

$$\frac{1}{\rho} = \frac{1}{\rho_f} .$$

For beams of arbitrary cross section, having an axis of symmetry, the curvature at fictitious elastic work in the elasto-plastic region is

$$\frac{1}{\rho_f} = \frac{M_f}{E_t I} ,$$

and the reduced fictitious moment is

$$M_f = \frac{\sigma_{s\,compr} I}{h_{elast.\,compr}} .$$

Since

$$M_{elast} = \sigma_{compr} W \quad \text{and} \quad W = \frac{I}{H'} ,$$

therefore,

$$M_f = \frac{M_{elast} H'}{h_{elast.\,compr}} . \qquad (8.83)$$

Here H' is the distance from the center of gravity of the cross section to the stretched end fiber.

Further, determination of deformations by the graphical-analytical method is similar to the solution obtained for a bar of a rectangular cross section. The coordinates of the boundaries of the parts of elastic and elasto-plastic work of material of the bar and the change in these coordinates with time are determined from the geometrical ratios depending on the diagram of the real moments.

Let us consider the determination of dimensions of the compressed part of the elastic core and of the coefficients μ_t for bars of cross section, other than rectangular.

The area of circular cross section is (Fig. 49a)

$$F = \pi r^2 .$$

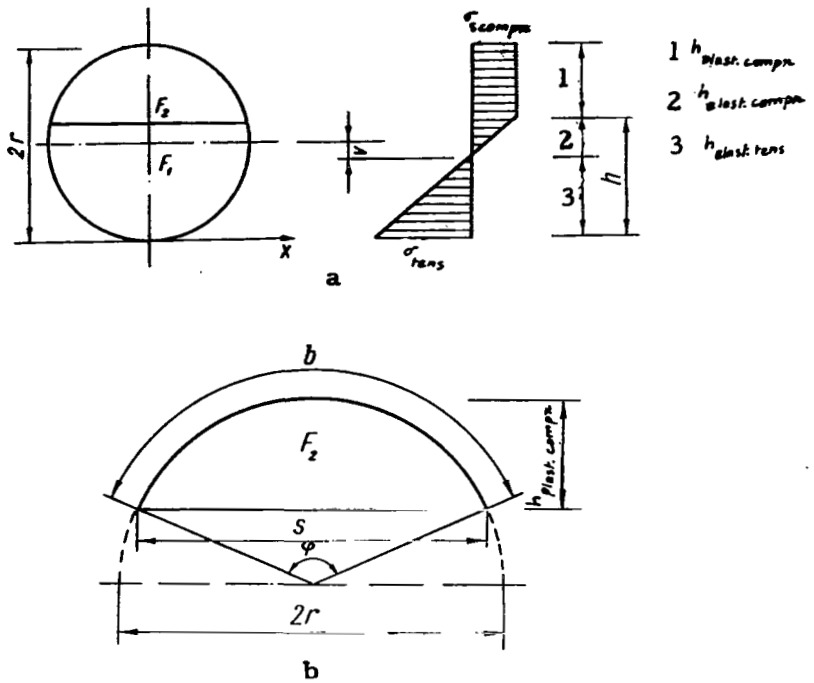


Fig. 49. Circular cross section of the bar.

Static moment of the section about the axis is

$$S_x = \pi r^3.$$

Considering the area of the plastic core as a segment of a circle (Fig. 49, b), we have

$$F_2 = \frac{rb + sh_{plast.compr} - sr}{2},$$

where b is the length of the arc, s is the length of the chord and $h_{plast.compr}$ is the height of the segment (height of the plastic core).

The area of the elastic core is

$$F_1 = \frac{2\pi r^2 - rb - sh_{plast.compr} + sr}{2}$$

Distance of the diameter from the center of gravity of the segement is

$$x_1 = \frac{s^3}{12F_2} .$$

Distance from the center of gravity of the plastic core to the stretched end fiber is

$$x_2 = r + \frac{s^3}{12F_2} .$$

Static moment of the plastic core about the x-axis is

$$S_{2x} = F_2 \cdot x_2 = F_2 r + \frac{s^3}{12} .$$

The Static moment of the elastic core about the x-axis is

$$S_{1x} = S_x - S_{2x} = F_1 r - \frac{s^3}{12} .$$

Moment of inertia of the elastic core about the x-axis is

$$I_{1x} = \frac{5\pi r^4}{4} - \frac{5r^2}{8}(rb - rs + h_{plast.\,compr}\, s) + \frac{s^3}{16}\left(h_{plast.\,compr} - \frac{11}{3} r\right).$$

All the formulas for determining the geometrical properties of the circular section contain unknown quantities like the length of the arc b, length of the chord s and height of plastic core $h_{plast.\,compr}$. According to Fig. 49, b, the length of the chord considered as a function of the height of the plastic core $h_{plast.\,compr}$ is

$$s = 2\sqrt{2h_{plast.\,compr}\, r - h^2_{plast\ compr}}$$

Exact value of the length of the arc is

$$b = \frac{\pi r \varphi^\circ}{180^\circ} ,$$

where φ° is the central angle.

The height of the plastic core is determined from the formula for an arbitrary type of section by putting the values of the geometrical characteristics

$$h_{plast.\,compr} = 2r - \frac{S_{1x}(\eta_t + 1)}{F_1(\eta_t + 1) - F} . \qquad (8.84)$$

Let us solve the equation (8.84) for η_t:

$$\eta_t = \frac{F(h_{plast.\,compr} - 2r)}{S_{1x} + F_1(h_{plast.\,compr} - 2r)} - 1. \qquad (8.85)$$

Since F_1 and S_{1x} can be expressed by the height of the plastic core $h_{plast.\,compr}$ by knowing the value of $h_{plast.\,compr}$ and taking the exact values of length of the arc, we obtain exact values of η_t corresponding to different values of $h_{plast.\,compr}$. Plotting the graph for the change in $h_{plast.\,compr}$ relative to η_t, we select the equation for the given curve. Such an approximate solution is completely admissible from the practical point of view.

The exact values of the coefficients μ_t depending upon η_t can be found in this manner. To obtain the values of the coefficient μ_t, we assume the value of $h_{plast.\,compr}$ and use the value of η_t obtained from formula (8.85). The equation for determining the values of coefficient μ_t has the form:

$$\mu_t = \frac{I_{1x}[F_1(\eta_t + 1) - \pi r^2]}{S_{1x}\pi r^3} - \frac{S_{1x}(\eta_t + 1) - \pi r^3}{\pi r^3}. \qquad (8.86)$$

Accurate values of η_t, μ_t, $h_{plast.\,compr}$, $h_{plast.\,tens}$ for a number of values of $h_{plast.\,compr}$ are presented in Table 12.

Table 12

$\frac{h_{plast.\,compr}}{r}$	η_t	μ_t	$\frac{h_{elast.\,compr}}{r}$	$\frac{h_{elast.\,tens}}{r}$
0	1.0	1.0	1.0	1.0
0.2	1.230	1.230	0.806	0.994
0.5	1.778	1.670	0.541	0.959
0.8	2.690	2.060	0.325	0.875
1.0	3.720	2.325	0.212	0.788
2.0	∞	4.0	0	0

Knowing the equation

$$h_{elast.\,compr} = \frac{r}{m_1 \eta_t^2 + m_2 \eta_t + m_3}$$

and using the values given in Table 12, an approximate formula for determining the dimension of the compressed part of the elastic core $h_{elast.\,compr}$ depending upon η_t is obtained:

$$h_{elast.\,compr} = \frac{r}{0.144\,\eta_t^2 + 0.690\,\eta_t + 0.166}. \qquad (8.87)$$

Let us write the equation

$$\mu_t = 4 - \frac{1}{n_1 \eta_t^2 + n_2 \eta_t + n_3}.$$

Here 4 is the ordinate equal to the limiting value of the coefficient μ_{lim} at $\eta_t \to \infty$.

Using the data from Table 12, we get the approximate formula for determining the coefficient μ_t depending upon η_t:

$$\mu_t = 4 - \frac{1}{0.185 + 0.162\,\eta_t - 0.014\,\eta_t^2}. \qquad (8.88)$$

Comparison of the exact values of $h_{elast.\,compr}$ and μ_t as a function of η_t with the approximate values shows that the approximate formulas lead to an error of less than 1 per cent.

Let us consider a semicircular section. There can be two cases of distribution of material in the section (Fig. 50). In using the calculation formulas, we shall resort to the same considerations as in the case of a circular section. Geometrical characteristics of the section for both the cases are presented in Table 13. Static moments and the moment of inertia are taken about the axis tangent to the stretched end fiber. The length of the chord in the first case is

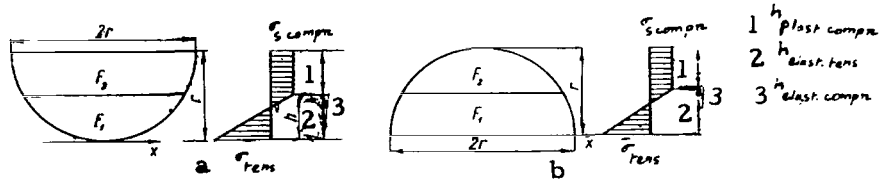


Fig. 50. Semicircular cross section of the bar.

$$s = 2\sqrt{2hr - h^2},$$

where h is the height of the elastic core;

In the second case

$$s = 2\sqrt{2h_{plast.compr}r - h^2_{plast.compr}},$$

where $h_{plast.\ compr}$ is the height of the plastic core.

Table 13

Geometrical characteristics	Case 1 (Fig. 50, a)	Case 2 (Fig. 50, b)
F	$\frac{\pi r^2}{2}$	$\frac{\pi r^2}{2}$
F_1	$\frac{rb - sr + sh}{2}$	$\frac{\pi r^2 - rb + sr - sh_{plast.\ compr}}{2}$
F_2	$\frac{\pi r^2 - rb + sr - sh}{2}$	$\frac{rb - sr + sh_{plast.\ compr}}{2}$
S_x	$0.2878\pi r^3$	$0.2122\pi r^3$
S_{1x}	$F_1 r - \frac{s^3}{12}$	$0.2122\pi r^3 - \frac{s}{12}$
S_{2x}	$0.2878\pi r^3 - F_1 r + \frac{s^3}{12}$	$\frac{s^3}{12}$
I_{1x}	$\frac{5r^2}{8}(rb - rs + hs) - \frac{s^3}{16}(h + \frac{5}{3}r)$	$\frac{\pi r^4}{8} - \frac{r^2}{8}(rb - rs + h_{plast.\ compr.})s - \frac{s^3}{16}(r - h_{plast.\ compr})$
W_{compr}	$0.306r^3$	$0.1908r^3$

The moment of resistance is taken about the compressed end fiber.

Assuming several values of $h_{plast.\ compr}$ or h and taking exact values of the length of the arc, a series of the corresponding exact values of the coefficients η_t and μ_t can be obtained.

By selecting the equations of the curves, we get approximate formulas for μ_t and $h_{elast.compr}$.

The size of the compressed part of elastic core $h_{elast.compr}$ as a function of the ratio η_t for the first case of the semicircular section is

$$h_{elast.compr} = \frac{r}{0.212\eta_t^2 + 1.210\eta_t + 0.325}, \quad (8.89)$$

and for the second case

$$h_{elast.compr} = \frac{r}{0.634\eta_t^2 + 1.310\eta_t + 0.427}. \quad (8.90)$$

Value of the coefficient μ_t as a function of the coefficient η_t for the first case of semicircular section is

$$\mu_t = 2.955 - \frac{10000}{8.2\eta_t^2 + 1015.8\eta_t + 3753}, \quad (8.91)$$

and for the second case

$$\mu_t = 3.5 - \frac{100}{0.365\eta_t^2 + 21.677\eta_t + 2.3826}, \quad (8.92)$$

On bending the bars of I-section, depending upon the ratio of the height of the shelf, thickness of the wall, and the ratio η_t of stresses in the stretched and compressed end fibers, two principal cases are distinguished.

Case I. Plastic deformations of compressed fibers are concentrated within the limits of one shelf of the section.

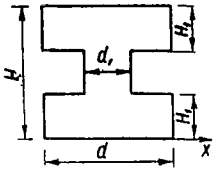
Case 2. Plastic deformations of compressed fibers cover one shelf and a part of the wall of the section.

Besides, there are two more cases, which are limiting relative to the principal calculated cases.

Case 1, a. Plastic deformations of compressed fibers completely cover one shelf of the section.

Case 2, a. Plastic deformations of compressed fibers completely cover one shelf and the wall of the section.

Table 14



$\alpha = \frac{d_1}{d}$; $\beta = \frac{H_1}{H}$

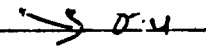
$F = (2\beta + \alpha - 2\alpha\beta)\, dH$

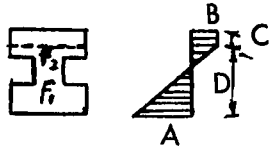
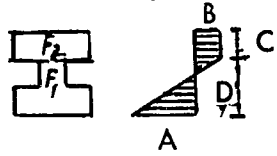
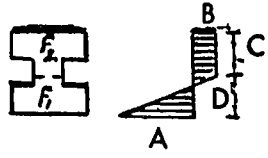
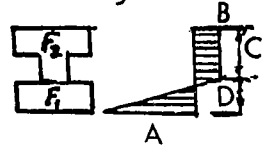
$S_x = (2\beta + \alpha - 2\alpha\beta)\, \frac{dH^2}{2}$

$W = [1 - (1-\alpha)(1-2\beta)^3]\, \frac{dH^2}{6}$

A – σ_{tens} C $h_{plast.compr}$

B – σ_{compr} D = h_{elast}



Geometrical characteristics	Case 1	Case 1, a	Case 2	Case 2, a
				
F_1	$dH\left(\alpha+\alpha\beta-2\alpha\beta-\frac{h_{plast.\,compr}}{H}\right)$	$dH(\alpha+\beta-2\alpha\beta)$	$dH\left(\beta-\alpha\beta+\alpha\frac{h_{elast}}{H}\right)$	$dH\beta$
F_2	$dh_{plast.\,compr}$	$dH\beta$	$dH\left(\beta+\alpha-\alpha\beta-\alpha\frac{h_{elast}}{H}\right)$	$dH(\beta+\alpha-2\alpha\beta)$
S_{1x}	$\frac{dH^2}{2}\left(\alpha+2\beta-2\alpha\beta-1+\frac{h^2_{elast}}{H^2}\right)$	$\frac{dH^2}{2}(\beta^2+\alpha-2\alpha\beta)$	$\frac{dH^2}{2}\left(\beta^2-\alpha\beta^2+\alpha\frac{h^2_{elast}}{H^2}\right)$	$\frac{dH^2}{2}\beta^2$
S_{2x}	$\frac{dH^2}{2}\left(1-\frac{h^2_{elast}}{H^2}\right)$	$\frac{dH^2}{2}\beta(2-\beta)$	$\frac{dH^2}{2}\left(\alpha+2\beta-2\alpha\beta-\alpha\beta^2-\beta^2-\alpha\frac{h^2_{elast}}{H^2}\right)$	$\frac{dH^2}{2}(\alpha-2\alpha\beta+2\beta-\beta^2)$
I_{1x}	$\frac{dh^3_{elast}}{3}$ $\frac{dH^3}{3}(1-\alpha)(1-3\beta+3\beta^2-2\beta^3)$	$\frac{dH^3}{3}(\beta^2+\alpha-2\alpha\beta^3-3\alpha\beta+3\alpha\beta^2)$	$\frac{dH^3}{3}\left(\beta^3-\alpha\beta^3+\alpha\frac{h^2_{elast}}{H^3}\right)$	$\frac{dH^3}{3}\beta^3$

The diagrams of normal stresses corresponding to the above mentioned cases are presented in Table 14. Formulas for determining the geometrical characteristics of the elastic and plastic cores depending upon the case under consideration are also given here. These formulas include the dimensions of the elastic and plastic core. In order to determine the dimensions of the elastic and plastic core, we use the formulas for the arbitrary cross section.

For case 1

$$h_{elast.\ compr} = \frac{H\{[1-\eta_t(1-\alpha)(2\beta-1)] + \sqrt{[1+(1-\alpha)(2\beta-1)][(1-\alpha)(2\beta-1)\eta_t^2+1]}\}}{(\eta_t+1)^2}; \quad (8.93)$$

for case 1, a, the height of the compressed part of the elastic core is

$$h_{elast.\ compr} = \frac{H[\beta^2+\alpha(1-2\beta)]}{2[\beta(\eta_t-1)+\alpha\eta_t(1-2\beta)]}; \quad (8.94)$$

for case 2

$$h_{elast.\ compr} = \frac{H\{[\alpha-(\eta_t+1)(1-\alpha)\beta] + \sqrt{[(\eta_t-1)(1-\alpha)\beta-\alpha]^2+\alpha\beta^2(1-\alpha)(\eta_t+1)^2}\}}{\alpha(\eta_t+1)^2}; \quad (8.95)$$

for case 2, a

$$h_{elast.\ compr} = \frac{H\beta}{2[\beta(\eta_t-1)-\alpha(1-2\beta)]}. \quad (8.96)$$

For case 1, the coefficient

$$\mu_t = \frac{4\left[\frac{h^3}{H^3}-(1-\alpha)(1-3\beta+3\beta^2-2\beta^3)\right]\left[\left(\alpha+2\beta-2\alpha\beta-\frac{h_{plast.\ compr}}{H}\right)\eta_t-\frac{h_{plast.\ compr}}{H}\right]}{\left(\alpha+2\beta-2\alpha\beta-1+\frac{h^2}{H^2}\right)[1-(1-\alpha)(1-2\beta)^3]} -$$

$$-\frac{3\left[\left(\alpha+2\beta-2\alpha\beta-1+\frac{h^2}{H^2}\right)\eta_t-1+\frac{h^2}{H^2}\right]}{[1-(1-\alpha)(1-2\beta)^3]}; \quad (8.97)$$

for case 1, a

$$\mu_t = \frac{4\{[\beta+\alpha(1-2\beta)]\eta_t-\beta\}[\beta^3-2\alpha\beta^3+2\alpha\beta^2-3\alpha\beta+\alpha]}{[\beta^2+\alpha(1-2\beta)][1-(1-\alpha)(1-2\beta)^3]} -$$

$$- \frac{3\{[\beta^2+\alpha(1-2\beta)]\eta_t-\beta(2-\beta)\}}{[1-(1-\alpha)(1-2\beta)^3]}; \qquad (8.98)$$

for case 2

$$\mu_t = \frac{4\left[\beta^3+\alpha\beta^3+\alpha\frac{h^3}{H^3}\right]\left[\left(\beta-\alpha\beta+\alpha\frac{h}{H}\right)\eta_t-\beta-\alpha+\alpha\beta+\alpha\frac{h}{H}\right]}{\left(\beta^2-\alpha\beta^2+\alpha\frac{H^2}{h^2}\right)[1-(1-\alpha)(1-2\beta)^3]} -$$

$$- \frac{3\left[\left(\beta^2-\alpha\beta^2+\alpha\frac{h^2}{H^2}\right)(\eta_t+1)-2\beta-\alpha+2\alpha\beta\right.}{[1-(1-\alpha)(1-2\beta)^3]}; \qquad (8.99)$$

and for case 2, a

$$\mu_t = \frac{4\beta[\beta\eta_t-\beta-\alpha(1-2\beta)]-3[\beta^2\eta_t-\beta(2-\beta)-\alpha(1-2\beta)]}{[1-(1-\alpha)(1-2\beta^3)]}. \qquad (8.100)$$

For calculating I-beams, it is essential to know the conditions of determination of the calculated case. In the case of I-beams with given dimensions, there exist the value η_{lim} at which plastic deformations of the compressed fibers develop within the limit of the compressed shelf and cover it completely. Here the value of η_{lim} depends only upon the ratio of the dimensions of the section and is a constant quantity while determining the size of the section. The value of η_{lim} can be obtained easily from the formula for determining the size of the compressed part of the elastic core $h_{elast.\,compr.}$ according to case 1, a. Comparing the values of η_{lim} with that of η_t, it can be ascertained, whether the plastic deformations of the compressed fibers will be concentrated only within the limits of the compressed zone, or they will include a part of the wall of the T-section besides the compressed shelf. If for a given material and size of the section, the value of η_t is less than η_{lim} for the same section, we shall use the formulas for

case 1. If the value of η_t is higher than that of η_{lim} we use the formulas for case 2.

Let us consider the determination of η_{lim} for I-section. Height of the plastic core is

$$h_{plast.\,compr} = H - h_{elast\,compr}\,(\eta_t + 1).$$

For case 1, a, $h_{plast.\,compr} = H_1 = \beta H$. Using the formula (8.94), we get the equation

$$\beta = 1 - \frac{[\beta^2 + \alpha(1-2\beta)](\eta_{lim}+1)}{2[\beta(\eta_{lim}-1) + \alpha\eta_{lim}(1-2\beta)]},$$

from which

$$\eta_{lim} = \frac{2\beta - \beta^2 + \alpha - 2\alpha\beta}{2\beta + \alpha - 4\alpha\beta - 3\beta^2 + 4\alpha\beta^2}. \qquad (8.101)$$

In an analogous way, it is possible to obtain fromulas for determining the deformation of bars of any cross section having axis of symmetry.

DEFORMABILITY DUE TO BENDING WITH CONSIDERATION OF PLASTIC DEFORMATIONS IN A COMPRESSED ZONE IN COURSE OF ENTIRE TIME OF ACTION OF LOAD

If for elasto-visco-plastic material

$$\sigma^{lim}_{str.\,compr} < \mu_{t\to\infty}\,\sigma_{l.l.n.\,compr},$$

the bending moment at time interval of action $t \longrightarrow 0$ is

$$M_{(t\to 0)} = \mu_{t\to 0}\,\sigma^{lim}_{str.\,compr}\,W,$$

and the growth of deformations in course of the entire time interval of test is accompanied by the formation of plastic deformations in the compressed zone of the beam. In the stretched part of the bar, the elastic work of material will exist only in such a case, when the constant bending moment does not exceed the limiting value.

At the time of loading ($t \longrightarrow 0$), there take place instantaneous elastic deformations in the stretched fibers and in parts

of the compressed fibers as well as plastic deformations in the remaining compressed fibers (within limits of the plastic core in the compressed zone of the bar). Thus, at the instant of loading, total deformations consist of elastic and plastic deformations.

After the application of the bending moment and formation of instantaneous deformations, elasto-viscous deformations begin to develop with time in the stretched fibers and part of the compressed fibers (elastic core of the section). In the remaining compressed fibers (plastic core of the section), plastic deformations develop with time. Total deformations at any moment of time consist of elasto-visco-plastic deformations, i. e., residual deformations arise immediately after loading and increase in their value in the course of an infinitely long time. The rate of their growth with time is related to the rate of growth of elasto-viscous deformations proceeding from the hypothesis of planar sections.

The redistribution of internal stresses takes place simultaneously with the development of deformations with time. This process is analogous to the one described earlier. The difference lies in the fact that it begins immediately after the application of the load, and not at the expiry of the limiting time interval $t_{s\,compr}$ since a part of the compressed fibers after loading is in a plastically deformed state.

The determination of curvature, angle of deflection of the sections and sags is analogous to that indicated earlier for the development of plastic deformations in the compressed part of the bar at time interval $t > t_{s\ compr}$.

During loading, the relative deformation of the stretched end fiber is

$$\varepsilon_{tens.\,inst} = \frac{\sigma_{tens}}{E_{inst}} ,$$

where the stresses in the stretched end fiber are determined with due consideration of the formation of plastic deformations in the compressed zone of the section.

Since the bending moment, in this case, is

$$M_x = \mu_{inst}\, \sigma^{lim}_{str.\,compr}\, W_{compr} ,$$

and

$$\eta_{inst} = \frac{\sigma_{tens}}{\sigma^{lim}_{str.compr}} ,$$

the stresses will be

$$\sigma_{tens} = \frac{M_x \eta_{inst}}{\mu_{inst} W_{compr}} . \qquad (8.102)$$

CHAPTER IX

CREEP DUE TO BENDING IN CASE OF DEVELOPMENT OF PLASTIC DEFORMATIONS IN COMPRESSED AND STRETCHED ZONES OF SECTION

VARIATION IN DIAGRAM OF NORMAL STRESSES WITH TIME

If during loading of a bendable bar, the continuously acting load gives rise to stresses in the end fibers higher than the limit of long-term resistance to tension, the resultant growth of fibrous deformations and total deformations of the bar with time can be divided into five periods.

In the first period arises the instantaneous elastic deformation, in the second the elasto-viscous deformation, and in the third the plastic deformation in the compressed zone of the bar subjected to fibrous stresses which change with time. The third period ends in the course of a time interval equal to the limiting time interval for the stretched end fiber, at the expiry of which the formation of plastic deformations in the stretched end fiber begins. Simultaneous development of plastic deformations in the compressed and stretched zones of the bar takes place in the fourth period. Here, the stresses change in the compressed as well as in the stretched zones of the bar. This period is characterized by a linear law of the formation of deformations with time under the action of a constant bending moment and ends at the expiry of the time interval of fracture of the stretched end fiber. In the fifth period, a quick separation of the parts of the bar takes place in the course of a short time interval.

Creep curves of pine wood, DSP-B plastic and KAST-V plastic glass subjected to pure bending are given in Fig. 51 and show that the fourth period, in which the formation of total deformations takes place with time according to a linear law, is very sharply demarcated.

The distribution of internal elastic forces in the section which began in the third period, continues with the formation of

plastic deformations in the stretched zone of the bar. Here, an increase in the height of plastic cores of the section of the bar takes place in the compressed and stretched zones with a corresponding decrease in the height of the elastic core in conformity with the law of planar sections and the law of linear distribution

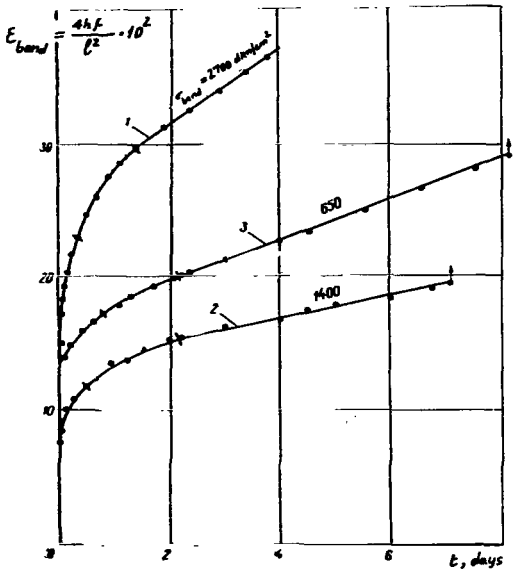


Fig. 51. Creep diagrams of DSP-B plastic (1), KAST-V plastic glass (2) and pine wood (3) subjected to bending.

of stresses along the height of the section in the elastic core. This process of redistribution of internal stresses along the height of the section takes place in the course of a limited time interval and ends when the stresses in the stretched end fiber attain the value of ultimate strength.

It is known that the development of plastic deformations with time, leading to the fracture of the bar, takes place with a constant rate at a constant value of the bending moment. The linear law of the growth with time has been established by experiments for sags. However, if the hypothesis of planar sections is correct, the linear law must also be true for the growth of fibrous deformations with time.

With the appearance of plastic deformations in the stretched end fibers, their further growth takes place with a constant rate attained at the time of their appearance. This constant rate

is, therefore, the limiting rate of development of elastic deformations in the stretched end fiber.

The appearance of plastic deformations along the height of the section is accompanied by redistribution of stresses with time. The constancy of the rate of growth of plastic fibrous deformations with time in the plastic core of the section, of the rate of growth of elastic deformations in the elastic core of the section, and consequently of the rate of growth of total deformations is observed during this process.

On the basis of the hypothesis of planar sections, if the diagram of the rate of development of deformations in fibers remains unchanged with time, the diagram of relative deformations for each time interval retains the linear form.

The diagram of normal stresses within the limits of the plastic core of the stretched zone as well as within the limits of the plastic core of the compressed zone of the section, has a curvilinear form (Fig. 52, a). Using the earlier mentioned simplified trigonal-trapezoidal form of the diagram of normal stresses (Fig. 52, b), it is possible to obtain simple analytical relations for the change in the carrying capacity and deformability of the bars subjected to bending when plastic deformations develop with time in the stretched and compressed zones of the section.

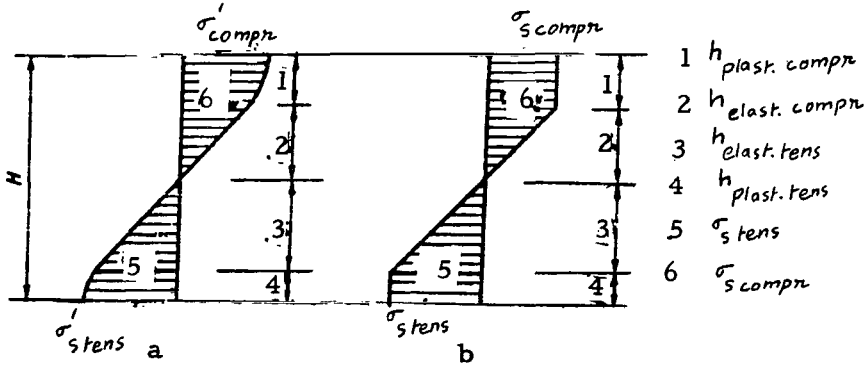


Fig. 52.Real (a) and simplified (b) diagrams of normal stresses.

Let us assume, as earlier, that the elastic limits of the compressed fibers vary with time according to the exponential law

$$\sigma_{scompr} = \frac{\sigma_{l.l.\,r.\,compr}}{1-(1-R_2^{compr})e^{-\alpha_{compr}\left(\frac{t_{scompr}}{t_{1,compr}}\right)^{c_{compr}}}}$$

Elastic limit of the stretched end fiber is determined from the formula

$$\sigma_{s\,tens} = \frac{\sigma_{e.l.\,tens}}{1-(1-k_2)e^{-\alpha_{tens}\left(\frac{t_{s\,tens}}{t_{tens}}\right)^{c_{tens}}}}.$$

When the time interval $t > t_{s\ tens}$, these stresses remain unchanged and give rise to plastic deformation in the stretched fibers which increases at a constant rate.

Let us determine the dimensions of the plastic and elastic cores along the height of the section of the bar. Since the form of the stress diagram (Fig. 53) is determined by four unknown dimensions of the core, it is necessary to formulate four equations for their determination. For this purpose, let us utilize the following conditions. Total height of the section is

$$H = h_{plast.\,compr} + h_{elast.\,compr} + h_{elast.\,tens} + h_{plast.\,tens} \quad . (9.1)$$

Let us denote the ratio of the stresses in the compressed end fiber to the stresses in the stretched end fiber at time t by η_t. According to Fig. 53,

$$\frac{\sigma_{s\,tens}}{\sigma_{s\,compr}} = \eta_t = \frac{h_{elast.\,tens}}{h_{elast.\,compr}} \qquad (9.2)$$

The third equation is the equation of the projection of internal elastic forces

$$\sum x = 0. \qquad (9.3)$$

The fourth equation may be obtained from the hypothesis of planar sections.

Fibrous deformation of the stretched end fiber at time interval $t > t_{s\ tens}$ attains values, corresponding to the stresses $\sigma_{s\ tens}$ which are taken to be constant within the limits of the plastic core of the stretched part of the section. Since the limiting elastic tensile deformation corresponds to these stresses, the fiber, at which the limiting elastic deformation is formed, is the boundary between the elastically and plastically deformed fibers. According to Fig. 53

$$\frac{\varepsilon_{total.\,tens}}{\varepsilon_{s\,tens}} = \frac{h_{elast.\,tens} + h_{elast.\,p'l.s.}}{h_{elast.\,tens}}. \qquad (9.4)$$

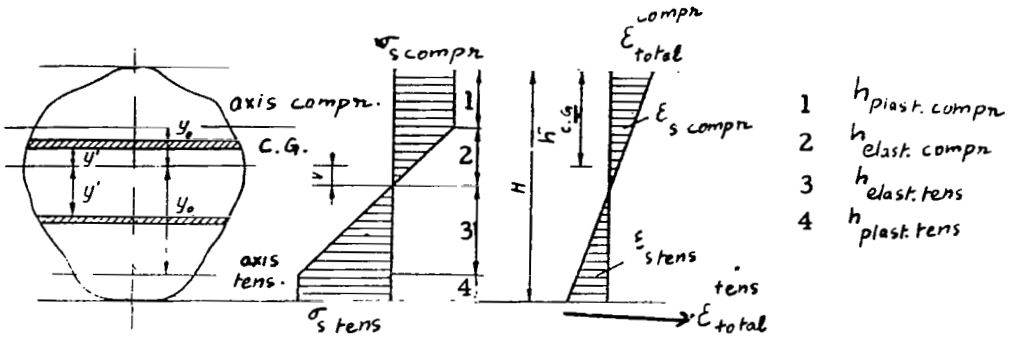


Fig. 53. Diagrams of normal stresses and relative fibrous deformations.

Deformation of the stretched end fiber $\varepsilon_{total\ tens}$ is equal to the sum of plastic and limiting elastic deformations. Let us denote the ratio of the total plastic deformation in the stretched end fiber to the limiting elastic deformation as

$$\frac{\varepsilon_{plast.\ tens}}{\varepsilon_{s\ tens}} = \frac{h_{plast.\ tens}}{h_{elast.\ tens}} = a_t. \qquad (9.5)$$

Using the formulas obtained earlier in the case of the axial action of the load, it may be put that in the case of

$$\varepsilon_{Plast.\ tens} = \frac{\sigma_{s\ tens}\, k\, \alpha_{tens}}{E_{prol}} \left(\frac{t_{s\ tens}}{t_{1\ tens}}\right)^{c_{tens}} e^{-\alpha_{tens}\left(\frac{t_{s\ tens}}{t_{1\ tens}}\right)^{c_{tens}}} \left(\frac{t}{t_{s\ tens}} - 1\right), \qquad (9.6)$$

$$\varepsilon_{s\ tens} = \frac{\sigma_{s\ tens}}{E_{prol}} \left[1 - k e^{-\alpha_{tens}\left(\frac{t_{s\ tens}}{t_{1\ tens}}\right)^{c_{tens}}}\right]. \qquad (9.7)$$

From formula (9.2) and Fig. 53,

$$\eta_t = \frac{\sigma_{s\ tens}}{\sigma_{s\ compr}} = \frac{\varepsilon_{s\ tens}}{\varepsilon_{s\ compr}}. \qquad (9.8)$$

Using the formula (9.7) and the formula for the determination of limiting elastic deformation due to axial compression

$$\varepsilon_{s\ compr} = \frac{\sigma_{s\ compr}}{E_{prol}} \left[1 - k e^{-\alpha_{compr}\left(\frac{t_{s\ compr}}{t_{1\ compr}}\right)^{c_{compr}}}\right], \qquad (9.9)$$

we get

$$\eta_t = \frac{\sigma_{s\,tens}\left(1-Re^{-\alpha_{tens}\left(\frac{t_{s\,tens}}{t_{1\,tens}}\right)^{c_{tens}}}\right)}{\sigma_{s\,compr}\left(1-Re^{-\alpha_{compr}\left(\frac{t_{s\,compr}}{t_{1\,compr}}\right)^{c_{compr}}}\right)}. \qquad (9.10)$$

Putting the values of $\varepsilon_{plast.\,tens}$ and $\varepsilon_{s\,tens}$ in the formula (9.5), we get

$$a_t = \frac{R\alpha_{tens}\left(\frac{t}{t_{s\,tens}}-1\right)\left(\frac{t_{s\,tens}}{t_{1\,tens}}\right)^{c_{tens}}}{e^{\alpha_{tens}\left(\frac{t_{s\,tens}}{t_{1\,tens}}\right)^{c_{tens}}}-R}. \qquad (9.11)$$

In order to determine the dimensions of elastic and plastic cores, we shall use the four equations mentioned above. From the condition (9.3)

$$-\sigma_{s\,compr}F_{compr.\,plast}+\sigma_{s\,tens}F_{tens.\,plast}-\int_{F_{elast.\,compr}}\sigma'_{compr}\,dF_{elast.\,compr}+$$

$$+\int_{F_{elast.\,tens}}\sigma'_{tens}\,dF_{elast.\,tens}=0, \qquad (9.12)$$

where $\sigma_{s\,compr}$ and $\sigma_{s\,tens}$ are stresses within the limits of the plastic cores of the compressed and stretched parts of the cross section; $F_{compr.\,plast}$ and $F_{tens.\,plast}$ are the areas of plastic core in the compressed and stretched parts of the cross section; σ'_{compr} and σ'_{tens} are stresses in the compressed and stretched parts of elastic core for the fiber, situated at a distance y' from the neutral axis.

Since, according to the stress diagram (see Fig. 53)

$$\sigma'_{compr}=\sigma_{s\,compr}\frac{y'}{h_{elast.\,compr}} \quad \text{and} \quad \sigma'_{tens}=\sigma_{s\,tens}\frac{y'}{h_{elast.\,tens}},$$

then

$$-\sigma_{scompr} F_{compr.plast} + \sigma_{stens} F_{tens.plast} -$$

$$- \frac{\sigma_{scompr}}{h_{elast.compr}} \int_{F_{elast.compr}} y' dF_{elast.compr} + \frac{\sigma_{stens}}{h_{elast.tens}} \int_{F_{elast.tens}} y' dF_{elast.tens} = 0,$$

as

$$\sigma_{stens} = \eta_t \sigma_{scompr},$$

therefore

$$-F_{compr.plast} + \eta_t F_{tens.plast} + \frac{1}{h_{elast.compr}} \left(\int_{F_{elast.tens}} y' dF_{elast.tens} - \right.$$

$$\left. - \int_{F_{elast.compr}} y' dF_{elast.compr} \right) = 0,$$

or

$$-F_{compr.plast} + \eta_t F_{tens.plast} + \frac{S_{elast}}{h_{elast.compr}} = 0, \qquad (9.13)$$

where S_{elast} is the statical moment of the elastic core about the neutral axis.

From equation (9.13), we get the height of the compressed part of the elastic core of the section

$$h_{elast.compr} = \frac{S_{elast}}{F_{compr.plast} - \eta_t F_{tens.plast}}. \qquad (9.14)$$

The statical moment of elastic core of the section about the central axis is equal to the statical moment of the plastic core about the same axis:

$$S_{elast.central} = S_{plast.central} \qquad (9.15)$$

The statical moment of the elastic core of the section about the central axis is

$$S_{elast.\,central} = S_{elast} + vF_{elast}, \qquad (9.16)$$

where v is the displacement of the neutral axis relative to the center of gravity of the section; F_{elast} is the area of the elastic core of the section.

From the equation (9.16), we get

$$S_{elast} = S_{elast.\,central} - vF_{elast} \qquad (9.17)$$

From the equation (9.14), we get

$$S_{elast} = h_{elast.\,compr}\left(F_{compr.\,plast} - \eta_t F_{tens.\,plast}\right). \qquad (9.18)$$

Equating (9.17) and (9.18), we obtain

$$S_{elast.\,central} - vF_{elast} = h_{elast.\,compr}\left(F_{compr.\,plast} - \eta_t F_{tens.\,plast}\right).$$

Replacing $S_{elast.\,central}$ from (9.16), we get

$$F_{plast.\,central} - vF_{elast} = h_{elast.\,compr}\left(F_{compr.\,plast} - \eta_t F_{tens.\,plast}\right). \qquad (9.19)$$

Since the distance from the center of gravity of the section to the base of the plastic core adjoining the elastic core of the compressed part of the section is

$$y_o = h_{elast.\,compr} - v,$$

and the distance from the center of gravity of the section to the base of the plastic core adjoining the elastic core of the stretched part of the section is

$$y_o' = h_{elast.\,tens} + v,$$

the equation (9.19) may be written as

$$S_{plast.\,central} - vF_{elast} = y_o F_{compr.\,plast} - y_o' F_{tens.\,plast} +$$

$$+ v\left(F_{tens.\,plast} + F_{compr.\,plast}\right). \qquad (9.20)$$

The area of the section is

$$F = F_{elast} + F_{tens.plast} + F_{compr.plast}.$$

Then, the equation (9.20) has the form

$$vF = S_{plast.central} + y_0' F_{tens.plast} - y_0 F_{compr.plast}. \quad (9.21)$$

Denoting the statical moment of plastic core of compressed part of the section about its base by $S_{b\ compr}$, and the statical moment of plastic core of stretched part of the section about its base by $S_{b\ tens}$, we get the statical moment of plastic core of section about the central axis

$$S_{plast.central} = \left(S_{base\ compr} + F_{compr.plast}\ y_0\right) - \left(S_{base\ tens} + F_{tens.plast}\ y_0'\right).$$

Putting the value of $S_{plast.central}$ in the equation (9.21), the displacement of the neutral axis v relative to the center of gravity is obtained as

$$v = \frac{S_{base\ compr} - S_{base\ tens}}{F}. \quad (9.22)$$

The distance from the center of gravity of the section to the stretched end fiber is

$$h_1 = \frac{S_x}{F}, \quad (9.23)$$

where S_x is the statical moment of the entire section about the axis, passing through the stretched end fiber.

Since

$$h_{elast.tens} + h_{plast.tens} = h_1 - v,$$

therefore

$$h_{elast.tens} + h_{plast.tens} = \frac{S_x}{F} - \frac{S_{base\ compr} - S_{base\ tens}}{F} \quad (9.24)$$

According to the equation (9.5)

$$h_{plast.tens} = h_{elast.tens}\ a_t$$

Then, by using the equation (9.24), the height of the stretched part of the elastic core is obtained as

$$h_{elast.\,tens} = \frac{S_x - S_{base\ compr} + S_{base\ tens}}{F(1+a_t)} \qquad (9.25)$$

and the height of plastic core in the stretched part of the section is

$$h_{plast.\,tens} = \frac{S_x - S_{base\ compr} + S_{base\ tens}}{F(1+a_t)}\, a_t \qquad (9.26)$$

Since

$$h_{elast.\,compr} = \frac{h_{elast.\,tens}}{\eta_t},$$

the height of the compressed part of the elastic core is

$$h_{elast.\,compr} = \frac{S_x - S_{base\ compr} + S_{base\ tens}}{F\eta_t(1+a_t)}. \qquad (9.27)$$

From the equation (9.1), the height of the plastic core in the compressed part of the section is obtained as

$$h_{plast.\,compr} = H - \frac{S_x - S_{base\ compr} + S_{base\ tens}}{F} \cdot \frac{1+\eta_t+\eta_t a_t}{(1+a_t)\eta_t}, \qquad (9.28)$$

where H is the total height of the section.

The bending moment, which is constant along the length of the bar in the given section, is equal to the moment of internal elastic forces in the cross section of the bar (see Fig. 53):

$$M = \sigma_{s\,compr} F_{plast.\,compr}(\zeta + h_{elast.\,compr}) + \sigma_{s\,tens} F_{tens.\,plast}(\zeta' + h_{elast.\,tens}) +$$

$$+ \frac{\sigma_{s\,compr}}{h_{plast.\,compr}} \int_{F_{compr.\,elast}} y'\,dF_{compr.\,elast} + \frac{\sigma_{s\,tens}}{h_{elast.\,tens}} \int_{F_{elast.\,tens}} y'\,dF_{tens.\,elast}$$

or

$$M = \sigma_{s\,compr} S^{n-n}_{compr.plast} + \sigma_{s\,tens} S^{n-n}_{tens.plast} + \sigma_{s\,compr} \frac{I^{n-n}_{elast.compr}}{h_{elast.compr}} +$$

$$+ \sigma_{s\,tens} \frac{I^{n-n}_{elast.tens}}{h_{elast.tens}}, \qquad (9.29)$$

where $S^{n-n}_{compr.plast}$ and $S^{n-n}_{tens.plast}$ are the statical moments of plastic cores of the compressed and stretched parts of the section about the neutral axis; $I^{n-n}_{elast.compr}$ and $I^{n-n}_{elast.tens}$ are the moments of inertia of the compressed and stretched parts of the elastic core of the section about the neutral axis.

Finally, the value of the bending moment, which is constant along the length of the bar in the given section, may be expressed as

$$M_x = \sigma_{s\,compr} \mu_t W_{tens}, \qquad (9.30)$$

where W_{tens} is the moment of resistance of the cross section about the stretched end fiber and μ_t is the coefficient taking into account the increase in the carrying capacity due to the development of plastic deformations in a part of a section as compared to the limiting elastic carrying capacity of the beam.

If the bending moment does not change with time, the value of coefficient μ_t must change with time, because the value of η_t and the dimensions of the plastic core in the compressed zone and that of the elastic cores in the compressed and stretched zones of the beam change with time.

The formula for determining the coefficient μ_t, changing with time, has the form:

$$\mu_t = \frac{S^{n-n}_{compr.plast} + \frac{I^{n-n}_{elast.compr}}{h_{elast.compr}} + \eta_t \left(S^{n-n}_{compr.tens} + \frac{I^{n-n}_{elast.tens}}{h_{elast.tens}} \right)}{W_{tens}} \qquad (9.31)$$

31)

The quantities $S^{n-n}_{compr.plast}$, $I^{n-n}_{elast.compr}$, $S^{n-n}_{compr.tens}$ and $h_{elast.tens}$ change with time. After some modification of (9.31), we get

$$\mu_t = \frac{I_{1x}[F_{1(x)}(\eta_t+1)-F]}{S_{1x}W_{compr}} - \frac{S_{1x}(\eta_t+1)-S_x}{W_{compr}}, \quad (9.32)$$

where all the geometrical properties of the section are taken about the axis passing through the stretched end fiber of the section.

Putting the values of η_t and μ_t in (9.30) and solving it with respect to stresses in the stretched end fiber σ_{stens}, their value may be obtained as

$$\eta_t = f(M_x) \quad (9.33)$$

and

$$a_t = f(M_x; \sigma_{scompr}) \quad (9.34)$$

and the expression, necessary for determining the bending deformations in the case of elasto-plastic region of work of the material can be obtained.

Let us consider some particular cases. Using the formulas obtained earlier and putting in them the values of the geometrical properties of the rectangular section, it is possible to determine the height of the elastic and plastic cores as well as the values of the coefficient μ_t.

Finally, for a rectangular section of the bar, the height of elastic and plastic cores are

$$h_{compr.plast} = H\left[1 - \frac{2(1+\eta_t+a_t\eta_t)}{(\eta_t+1)(\eta_t+1+2a_t\eta_t)}\right], \quad (9.35)$$

$$h_{compr.plast} = \frac{2H}{(\eta_t+1)(\eta_t+1+2a_t\eta_t)}, \quad (9.36)$$

$$h_{tens.elast} = \frac{2H\eta_t}{(\eta_t+1)(\eta_t+1+2a_t\eta_t)}, \quad (9.37)$$

$$h_{tens.plast} = \frac{2a_t\eta_t H}{(\eta_t+1)(\eta_t+1+2a_t\eta_t)} \quad (9.38)$$

Displacement of the neutral axis relative to the center of gravity of the section is

$$v = \frac{H}{2}\left[1 - \frac{4\eta_t(1+a_t)}{(\eta_t+1)(\eta_t+1+2a_t\eta_t)}\right]. \qquad (9.39)$$

The coefficient

$$\mu_t = 3 - \frac{4\left[(\eta_t+1)^2 + 3a_t\eta_t(1+\eta_t) + 3a_t^2\eta_t^2\right]}{(\eta_t+1)(\eta_t+1+2a_t\eta_t)^2}. \qquad (9.40)$$

The value of the coefficient μ_t depends on the quantities η_t and a_t, which change with time under the action of a constant load.

Let us determine the dimensions of elastic and plastic cores of the section and the coefficient μ_t for a circular cross section and also the necessary geometrical properties of the section.

The area of the cross section (Fig. 54) is

$$F = \pi r^2.$$

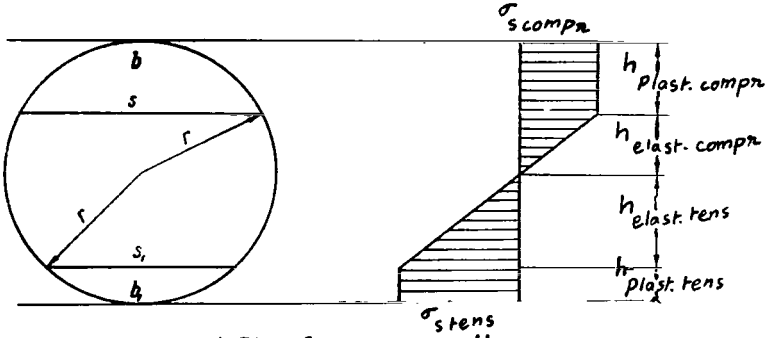


Fig. 54. Circular cross section.

The statical moment of the section about the x-axis is

$$S_x = \pi r^3.$$

Moment of resistance is

$$W = \frac{\pi r^3}{4}.$$

The statical moment of the plastic core in the compressed zone about its base is

$$S_{base\ compr} = F_{plast.\ compr}\, y_1,$$

where y_1 is the distance from the center of gravity of the plastic core of the section in the compressed zone to its base.

The area of the plastic core in the compressed zone (see Fig. 54) is

$$F_{plast.\ compr} = \frac{rb + sh_{plast.\ compr} - sr}{2},$$

where b is the length of the arc within the limits of the plastic core of the compressed zone; s is the length of the chord in the same place and $h_{plast.\ compr}$ is the height of the segment (height of the plastic core).

The quantity

$$y_1 = \frac{s^3}{12F_{plast.\ compr}} - \sqrt{r^2 - \frac{s^2}{4}}.$$

Putting the value of $F_{plast.\ compr}$ and y_1, we get

$$S_{base\ compr} = \frac{s^3}{12} - \frac{\sqrt{4r^2 - s^2}\,(rb + sh_{plast.\ compr} - sr)}{4}.$$

Similarly, we obtain the statical moment of the plastic core in the stretched zone about its base

$$S_{base.\ tens} = -\frac{s_1^3}{12} + \frac{\sqrt{4r^2 - s_1^2}\,(rb + s_1 h_{plast.\ tens} - s_1 r)}{4}.$$

Here s_1 and b_1 are the respective lengths of arc and chord within the limits of plastic core in the stretched zone.

Using the values of the geometrical properties for the circular section, the height of the plastic core of the section in the stretched zone may be obtained as

$$h_{plast.\ tens} = \frac{a}{(1+a)\pi r^2}\left[\pi r^3 - \frac{s^3}{12} + \frac{\sqrt{4r^2 - s^2}\,(rb + sh_{plast.\ compr} - sr)}{4} - \right.$$

$$\left. - \frac{s_1^3}{12} + \frac{\sqrt{4r^2 - s_1^2}\,(rb_1 + s_1 h_{plast.\ tens} - s_1 r)}{4}\right]. \qquad (9.41)$$

According to Fig. 54, the length of the chord in the plastic core of the compressed zone is

$$s = \sqrt{2h_{Plast.\,compr}\,r - h^2_{Plast.\,compr}},$$

and in the plastic core of the stretched zone is

$$s_1 = \sqrt{2h_{Plast.\,tens}\,r - h^2_{Plast.\,tens}}.$$

Let us take exact values of the length of the arc

$$b = \frac{\pi r \varphi^\circ}{180^\circ}.$$

Assuming different values of a, it is possible to determine the change in the height of the plastic core in the stretched zone of the section and to find out the dimensions of the plastic core in the compressed zone and of the elastic cores of the section in the compressed and stretched zones.

Using the above mentioned method, it is possible to determine the values of the coefficient μ_t for beams with circular cross section, as well as for beams with semicircular cross section.

As an illustration of the determination of the carrying capacity of I-beam, let us consider the case, when plastic deformations cover completely the upper and the lower shelves, and the wall works elastically (Fig. 55).

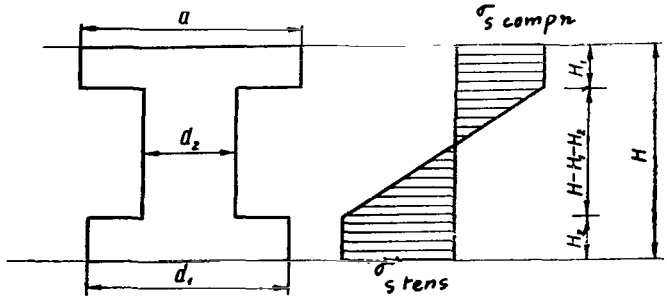


Fig. 55.I-shaped cross section.

The height of the plastic core in the compressed zone is equal to the height of the upper shelf of the I-section

$$h_{Plast.\,compr} = H_1,$$

and the height of the plastic core in the stretched zone is equal to the height of the lower shelf of the I-section

$$h_{plast.\ tens} = H_2 .$$

In order to determine the dimensions of the elastic cores in the compressed and stretched zones, the formulas for arbitrary section or the condition of equilibrium $\Sigma x = 0$ may be used.

Formulating the equation of projections for x-axis, we get

$$\sigma_{compr} d H_1 + \frac{\sigma_{compr} d_1 h_{elast.\ compr}}{2} - \frac{\sigma_{tens} d_1 h_{elast.\ tens}}{2} - \sigma_{tens} d_2 H_2 = 0 \qquad (9.42)$$

Taking $\frac{\sigma_{tens}}{\sigma_{compr}} = \frac{h_{elast.\ tens}}{h_{elast.\ compr}} = \eta_t$ and solving the equation (9.42) with respect to $h_{elast.\ compr}$, we get the height of the elastic core in the compressed zone of the section

$$h_{elast.\ compr} = \frac{2(dH_1 - \eta_t d_2 H_2)}{d_1(\eta_t^2 - 1)} \qquad (9.43)$$

and the height of the elastic core in the stretched zone of the section

$$h_{elast.\ tens} = \frac{2\eta_t (dH_1 - \eta_t d_2 H_2)}{d_1(\eta_t^2 - 1)} \qquad (9.44)$$

The coefficient μ_t may be obtained from the equilibrium condition

$$\Sigma M_{int} = \mu_t \sigma_{compr} W_{compr} \qquad (9.45)$$

According to Fig. 55, the moment of internal forces about the extreme stretched fiber is

$$\Sigma M_{int} = \sigma_{compr} d H_1 \left(H - \frac{H_1}{2}\right) + \frac{\sigma_{compr} d_1 h_{elast.\ compr}}{2} \left(\frac{2h_{elast.\ compr}}{3} + h_{elast.\ tens} + H_2\right) -$$

$$- \frac{\sigma_{tens} d_1 h_{elast.\ tens}}{2} \left(\frac{h_{elast.\ tens}}{3} + H_2\right) - \frac{\sigma_{tens} d_2 H_2^2}{2} . \qquad (9.46)$$

Substituting $\sigma_{tens} = \eta_t \sigma_{compr}$, putting the values of $h_{elast.\,tens}$ and $h_{elast.\,compr}$, and equating ΣM_{int} according to the formulas (9.45) and (9.46), after transformation we get

$$\mu_t = \frac{\frac{d}{2}(2HH_1 - H_1^2 - 2H_1H_2) + \frac{d_2 H_2^2 \eta_t}{2} + \frac{2[2+\eta_t(1-\eta_t)](dH_1 - \eta_t d_2 H_2^2)}{3d_1(\eta_t - 1)^2(\eta_t + 1)}}{W_{compr}}. \quad (9.47)$$

Similarly, it is also possible to obtain formulas for other cases of distribution of stresses in I-beams, T-beams, box-shaped beams and beams of other cross sections.

DIFFERENTIAL EQUATION OF BENT AXIS OF A BAR

Increase in plastic deformations in the stretched zone of the section of the bar with continuing increase in the compressed zone leads to the result, that the differential equation of the bent axis, applicable for determining deformations of the elastic region of the bar and of the part subjected to the development of plastic deformations in the compressed region of the section, is not applicable here. For the part of the bar in which plastic deformation has been formed in the compressed and stretched zones of the section, it is essential to obtain a differential equation which should take into consideration the plastic deformation in the stretched zone of the bar.

Thus, in order to determine sag and angle of deflection, it is necessary to have three differential equations of the bent axis of the bar.

For the elastically working part of the bar, the differential equation

$$E_t I \frac{d^2y}{dx^2} = M_{(x)},$$

holds good, and for the part of the bar, at which plastic deformations develop only in the compressed zone of the section, the equation

$$\frac{d^2y}{dx^2} = \frac{\sigma_{scompr\,t}}{h_{elast.\,compr} E_t}.$$

is applicable.

Let us find out the differential equation for the part of the bar, in which plastic deformations develop in the compressed, as well as in the stretched zones of the section. Considering that the diagram of normal stresses has a trigonal-trapezoidal form in the compressed and stretched zones of the section and the hypothesis of planar section is correct, the expression for the curvature may be written as

$$\frac{1}{\rho_t} = \frac{\varepsilon_{compr.total} + \varepsilon_{tens.total}}{H}, \qquad (9.48)$$

where $\varepsilon_{compr.total}$ and $\varepsilon_{tens.total}$ are the relative deformations of the compressed and stretched end fibers at the time interval $t > t_{s\,tens}$.

According to Fig. 53, the curvature may be expressed as the ratio of the sum of the limiting elastic deformations of the compressed and the stretched fibers at the boundaries of the elastic core for the given time interval $t > t_{s\,tens}$ to the height of the elastic core:

$$\frac{1}{\rho_t} = \frac{\varepsilon_{s\,compr} + \varepsilon_{s\,tens}}{h_{elast.compr} + h_{elast.tens}} \qquad (9.49)$$

or as the ratio of the limiting elastic deformation of the compressed fiber at the boundary of the plastic core in the compressed zone to the height of the compressed part of the elastic core:

$$\frac{1}{\rho_t} = \frac{\varepsilon_{s\,compr}}{h_{elast.compr}}, \qquad (9.50)$$

or as the ratio of the limiting elastic deformation of the stretched fiber at the boundary of the plastic core in the stretched zone to the height of the stretched part of the elastic core:

$$\frac{1}{\rho_t} = \frac{\varepsilon_{s\,tens}}{h_{elast.tens}}. \qquad (9.51)$$

The relative limiting elastic deformation for the given time interval $t > t_{s\,tens}$ is

$$\varepsilon_{s\,tens} = \frac{\sigma_{s\,tens}}{E_t}. \qquad (9.52)$$

Putting the value of $\varepsilon_{s\,tens}$ in the formula (9.51), we get

$$\frac{1}{\rho_t} = \frac{\sigma_{s\,tens}}{h_{elast.tens} E_t} \qquad (9.53)$$

Since

$$\frac{\sigma_{s\,tens}}{\sigma_{s\,compr}} = \eta_t \quad \text{and} \quad \frac{h_{elast.\,tens}}{h_{elast.\,compr}} = \eta_t \ ,$$

the curvature is

$$\frac{1}{\rho_t} = \frac{\sigma_{s\,compr}}{h_{elast.\,compr} E_t} \ , \tag{9.54}$$

and the differential equation of the bent axis of the bar is

$$y'' = \frac{\sigma_{s\,compr}}{h_{elast.\,compr} E_t} \ . \tag{9.55}$$

Expressing the dimension of the compressed part of the elastic core $h_{elast.\,compr}$ as a function of the bending moment $M_{(x)}$ and applying ordinary methods of integrating differential equations, it is possible to obtain the values of angles of deflection and sags, as well as their change with time during bending of the bar in case of development of plastic deformations in the compressed and stretched zones of the beam.

Let us determine the deformation of bars with rectangular section in the case of a constant value of external load acting for a time interval $t > t_{s\,tens}$.

Since the bending moment $M_{(x)}$ along the length of the beam is variable (except for pure bending), the value of η_t and a_t is obtained as functions of $M_{(x)}$.

When the time interval $t > t_{s\,tens}$, the bending moment in the section x of the part of the bar with plastic deformations in the compressed and stretched zones, according to the diagram of normal stresses, is

$$M_{(x)} = \mu_t \sigma_{s\,compr} W,$$

where

$$\mu_t = 3 - \frac{4\left[(\eta_t+1)^2 + 3a_t\eta_t(1+\eta_t) + 3a_t^2\eta_t^2\right]}{(\eta_t+1)(\eta_t+1+a_t+\eta_t)^2} \ .$$

Assuming that the limiting elastic bending moment is

$$M_{elast} = \sigma_{s\,compr} W, \tag{9.56}$$

and the limiting elasto-plastic bending moment, at which plastic deformations develop only in the compressed zone of the section of the bar, is

$$M_{Plast.compr} = M_{elast} \frac{3\eta_t - 1}{\eta_t + 1}, \qquad (9.57)$$

and putting the value of $M_{plast.\,compr}$ in the formula (9.56) and solving it with respect to coefficient a_t, we get the value of the latter as a function of the acting and the limiting bending moment $M_{(x)}$ and $M_{plast.\,compr}$:

$$a_t = f(M_{(x)}, M_{Plast.\,compr}). \qquad (9.58)$$

Let us simplify the expression for the coefficient μ_t by neglecting the terms of the equation, containing the square of coefficient a_t, since the coefficient a_t varies, in practice, within the range of 0 to 0.3. Such simplification gives not more than 5 percent error in the values of μ_t in all the cases met with in practice.

Thus, the approximate expression for the coefficient μ_t may be written as

$$\mu_t = 3 - \frac{4(1+\eta_t+3a_t\eta_t)}{(1+\eta_t)(1+\eta_t+4a_t\eta_t)}. \qquad (9.59)$$

Solving the equation (9.59) with respect to a_t, we get

$$a_t = \frac{(1+\eta_t)[1-3\eta_t+(1+\eta_t)\mu_t]}{4\eta_t[3\eta_t-(1+\eta_t)\mu_t]}, \qquad (9.60)$$

where

$$\mu_t = \frac{M_{(x)}}{\sigma_{scompr} W_{compr}} = \frac{M_{(x)}}{M_{elast}}.$$

For any time interval $t > t_{s\,tens}$, the value of the bending moment $M_{(x)}$ in the portion of the bar with developing plastic deformations in the compressed and stretched zones depends (at constant value of the coefficient η_t for the given time interval) only on the coefficient a_t.

Using the formula (9.35), in which the value of the coefficient a_t (9.60) is put, we get an expression for the height of the compressed part of the elastic core

$$h_{elast.\,compr} = \frac{4H\left[3\eta_t - (1+\eta_t)\mu_t\right]}{(1+\eta_t)^2\left[1+3\eta_t-(1+\eta_t)\mu_t\right]} . \quad (9.61)$$

Putting the value of $h_{elast.\,compr}$ and substituting μ_t by the ratio of the bending moment in the given section to the limiting elastic moment, we get the differential equation of the bent axis

$$y'' = \frac{M_{elast}(1+\eta_t)^2\left[1+3\eta_t-(1+\eta_t)\frac{M_{(x)}}{M_{elast}}\right]}{E_t W 4H\left[3\eta_t-(1+\eta_t)\frac{M_{(x)}}{M_{elast}}\right]}$$

or

$$y'' = \frac{M_{elast}(1+\eta_t)^2\left[1+3\eta_t-(1+\eta_t)\frac{M_{(x)}}{M_{elast}}\right]}{8E_t I\left[3\eta_t-(1+\eta_t)\frac{M_{(x)}}{M_{elast}}\right]} . \quad (9.62)$$

This equation is true for the part of the bar, in which plastic deformations have developed in the compressed and stretched zones of the bar under the action of a constant load.

For the part of the bar acting only elastically in the compressed and stretched zones, an ordinary differential equation

$$y'' = \frac{M_x}{E_t I} ,$$

is used, and for the part of the beam in which plastic deformations develop only in the compressed zone, the equation

$$y'' = \frac{4M_{elast}}{E_t I\left(3-\frac{M_x}{M_{elast}}\right)^2} .$$

is used.

Determination of sags and angles of deflection along the entire length of the bar requires integration of all the above-mentioned three differential equations.

In the analytical method for determining deformations, it is essential to find out the values of six constants of integration.

Two constants are obtained from ordinary conditions. The rest of the constants of integration are determined from additional conditions, based on the consideration, that sags and angles of deflection are equal at the boundary between the elastic and elasto-plastic parts when plastic deformations develop only in the compressed zone, as well as at the boundary between the elasto-plastic part where plastic deformations develop only in the compressed part and the elasto-plastic part where plastic deformations develop both in the compressed and stretched zones:

$$\left.\begin{aligned} x = x_a\,,\ y'_{elast} = y'_{xa}\,,\ y_{elast} = y_{xa}\,;\\ x = x_b\,,\ y'_{xb} = y'_{xa}\,,\ y_{xb} = y_{xa}\,. \end{aligned}\right\} \qquad (9.63)$$

The coordinate x_a determines the boundary between the elastic and elasto-plastic parts in case of development of plastic deformations only in the compressed zone of the parts of the bar; x_b determines the boundary between the elasto-plastic part where plastic deformations develop only in the compressed zone and the part where plastic deformations develop in the compressed as well as in the stretched zones of the bar.

In order to determine deformations, it is also possible to use the graphical-analytical method discussed earlier in the case of bending.

In order to find out the fictitious bending moment in the part of the elasto-plastically working bar, the two-sided trigonal-trapezoidal diagram of normal stresses in each section is replaced by a fictitious diagram for a fictitious elastic bar, and then the reduced moments are obtained and the reduced projection diagram of bending moments is plotted, considering the bar as an elastic one, the load here differs from the real one. The condition of equality of sags and angles of deflection of the sections of the real and fictitious bars must be maintained here. For this, let us assume that the radii of curvature of the real and fictitious bars are equal. As a result, we get the fictitious bending moment in the part of the bar, within the limits of which plastic deformations develop in the compressed and stretched zones of the section:

$$(M_x)_f = \frac{M_{elast}(1+\eta_t)^2\left[1+3\eta_t-(1+\eta_t)\frac{M_x}{M_{elast}}\right]}{8\left[3\eta_t-(1+\eta_t)\frac{M_x}{M_{elast}}\right]}. \qquad (9.64)$$

Thus, in the determination of deformations by the graphical-analytical method, the bending moment diagram consists of three sections. In the first section, within the limit of which the material works elastically, the bending moment diagram is plotted by ordinary methods. In the second section, in which plastic deformations develop only in the compressed zone of the bar, the reduced bending moment diagram is plotted according to the formulas, obtained earlier in Chapter VIII. In the third section, where plastic deformations develop in the compressed and stretched zones of the bar, the reduced bending moment diagram is plotted according to the formula (9.64).

LIMITING CARRYING CAPACITY OF BARS SUBJECTED TO BENDING AND ITS VARIATION WITH TIME

In the case of bending of bars, the limiting carrying capacity is determined in terms of maximum breaking bending moment. The value of the breaking bending moment changes with time depending on the change in the value of limiting carrying capacity.

Under the action of a constant bending load, the maximum bending moment, depending upon its magnitude, may or may not give rise to the fracture of the bar in the course of the final time interval. Thus, there must exist a limiting value of the bending moment under the action of which fracture of the bar will take place at an infinitely long interval of time.

It is shown by experiments that the curves of prolonged resistance at bending, plotted in the coordinates "breaking bending moment M_{fn} and time till fracture t_{fn} ", have the general form of asymptotic curves.

Curves of prolonged resistance in the case of axial compression and tension are generally plotted in the coordinates "breaking stresses and time till fracture". The ordinate of the asymptote to the prolonged resistance curve at axial compression

or tension determines the value of the stresses equal to the limit of long-term resistance.

Since, while bending bars of different cross sections, the value of the fictitious stresses in the end fibers change depending upon the form of the section, it is reasonable to plot the prolonged resistance curve of bending not according to fictitious stresses in the end fibers, but according to limiting carrying capacity of the bar, which is determined by the breaking moment.

The ordinate of the asymptote to the prolonged resistance curve of bending determines the limit of long-term resistance as the minimum limiting carrying capacity of the bar at infinitely long time of action of a constant load.

The limiting carrying capacity in the case of bending is determined by stresses, attaining the value of ultimate strength, in the stretched end fiber in the corresponding time interval.

Thus, the time till fracture of the bar subjected to bending is equal to the time till fracture due to axial tension and may be determined from the prolonged resistance curve of tension.

The value of stresses in the compressed end fiber as well as within the limits of the plastic core of the compressed zone is determined from the relation between elastic limit and time, which is applicable to the case of compression.

Having the curve of prolonged resistance for axial tension and the curve showing the change in the elastic limit for axial compression, the prolonged resistance curve of bending may be plotted by using the stress diagram in the limiting state. In the case of bending, the diagram of normal stresses in the limiting state has the form shown in Fig. 56, a. Since fracture takes place after a lapse of time $t_{fr.tens}$, the stresses in the plastic core of the compressed part of the section are equal to the elastic limit $\sigma_{s\,compr}$ for axial compression for the same interval $t_{fr.tens}$, and in the plastic core of the stretched part of the section they are equal to the ultimate strength $\sigma_{fr.tens}$. At the boundary of the elastically stressed part of the section, stresses vary from $\sigma_{s\,compr}$ to $\sigma_{fr.tens}$ according to a linear law.

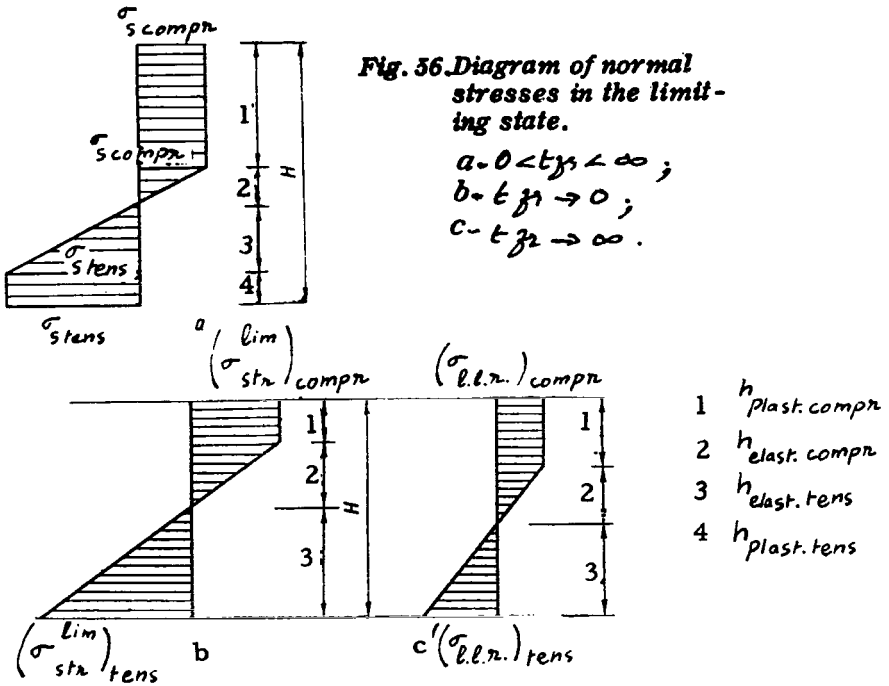


Fig. 56. Diagram of normal stresses in the limiting state.

a - $0 < t_{fr} < \infty$;
b - $t_{fr} \to 0$;
c - $t_{fr} \to \infty$.

Let us determine the limiting carrying capacity of a bar of an arbitrary section.

The value of the coefficient μ in the limiting state may be obtained from the formula (9.32), by putting in it the coefficients η_t and a_t and the geometrical properties of the section in the limiting state.

The coefficient $\eta_{t\,fr}$ in the limiting state is

$$\eta_{t\,fr} = \frac{\sigma_{fr.tens}}{\sigma_{s\,compr}}. \qquad (9.65)$$

In the case of fracture of the bar, fibrous deformation in the stretched end fiber attains the value of the breaking deformation $\varepsilon_{fr.tens}$, corresponding to the breaking stresses $\sigma_{fr.tens}$, which are constant within the limits of the plastic core of the stretched part of the section and give rise to a limiting elastic deformation in the fiber at the boundary between the elastic and the plastic core of the section.

Proceeding from the hypothesis of planar sections, it is possible to write

$$\frac{\varepsilon_{fr.tens}}{\varepsilon_{s\,tens}} = \frac{h_{elast.tens} + h_{plast.tens}}{h_{elast.tens}}. \qquad (9.66)$$

Since deformation ε_{fr} is equal to the sum of plastic and limiting elastic deformations

$$\varepsilon_{fr} = \varepsilon_{plast.tens} + \varepsilon_{s\,tens},$$

then

$$\frac{\varepsilon_{plast.tens}}{\varepsilon_{s\,tens}} = \frac{\varepsilon_{plast.tens}}{h_{elast.tens}} = a_{t_{fr}}. \qquad (9.67)$$

Using the formula (9.11), in which t is replaced by $t_{fr.tens}$, we get the coefficient

$$a_{t_{fr}} = \frac{k\alpha_{tens}\left(\frac{t_{fr.tens}}{t_{s\,tens}} - 1\right)\left(\frac{t_{s\,tens}}{t_{1\,tens}}\right)^{c_{tens}}}{e^{\alpha_{tens}\left(\frac{t_{s\,tens}}{t_{1\,tens}}\right)^{c_{tens}}} - k}. \qquad (9.68)$$

The geometrical characteristics of the section and the value of the coefficient $\mu_{t_{fr}}$ are determined from the formulas obtained earlier, by substituting the limiting values of the coefficients $\eta_{t_{fr}}$ and $a_{t_{fr}}$.

The value of coefficient $\eta_{t\,fr}$ changes with time according to the asymptotic curve from $\eta_{t\,fr} = \frac{(\sigma_{str}^{lim})_{tens}}{(\sigma_{str}^{lim})_{compr}}$ at $t_{fr} \to 0$ to $\eta_{t\,fr} = \frac{(\sigma_{l.l.n.})_{tens}}{(\sigma_{l.l.n.})_{compr}}$ at $t_{fr} \to \infty$.

The coefficient $a_{t\,fr}$ characterizes the plasticity of the material subjected to tension. Since plastic deformations are absent in the stretched fibers during the time of action of the breaking load $t_{fr} \to 0$ and $t_{fr} \to \infty$, the value of coefficient $a_{t\,fr}$ is equal to zero in both the cases. With the increase in the time interval t_{fr}, the value of the coefficient $a_{t_{fr}}$ increases initially, attains maximum, and then decreases till it is zero.

The limiting diagrams of normal stresses at $t_{fr} \to 0$ and $t_{fr} \to \infty$ have the form shown in Fig. 56, b and c.

For rectangular cross section at $t_{fr} \to 0$, the coefficient is

$$\mu_{\substack{t \to 0 \\ fr}} = 3 - \frac{4}{\eta_{\substack{t \to 0 \\ fr}} + 1}, \tag{9.69}$$

and at $t_{fr} \to \infty$

$$\mu_{\substack{t \to \infty \\ fr}} = 3 - \frac{4}{\eta_{\substack{t \to \infty \\ fr}} + 1}. \tag{9.70}$$

Putting the values of $\eta_{t_{fr}} \to 0$ and $\eta_{t_{fr}} \to \infty$, we get

$$\mu_{\substack{t \to 0 \\ fr}} = \frac{3\left(\sigma_{str}^{lim}\right)_{tens} - \left(\sigma_{str}^{lim}\right)_{compr}}{\left(\sigma_{str}^{lim}\right)_{tens} + \left(\sigma_{str}^{lim}\right)_{compr}}, \tag{9.71}$$

$$\mu_{\substack{t \to \infty \\ fr}} = \frac{3\left(\sigma_{l.l.r.}\right)_{tens} - \left(\sigma_{l.l.r.}\right)_{compr}}{\left(\sigma_{l.l.r.}\right)_{tens} + \left(\sigma_{l.l.r.}\right)_{compr}}. \tag{9.72}$$

Limiting bending moment at time $t_{fr} \to 0$ and $t_{fr} \to \infty$ is

$$M_{\substack{t \to 0 \\ fr}} = \frac{3\left(\sigma_{str}^{lim}\right)_{tens} - \left(\sigma_{str}^{lim}\right)_{compr}}{\left(\sigma_{str}^{lim}\right)_{tens} + \left(\sigma_{str}^{lim}\right)_{compr}} \left(\sigma_{str}^{lim}\right)_{compr} W, \tag{9.73}$$

$$M_{\substack{t \to \infty \\ fr}} = \frac{3\left(\sigma_{l.l.r.}\right)_{tens} - \left(\sigma_{l.l.r.}\right)_{compr}}{\left(\sigma_{l.l.r.}\right)_{tens} + \left(\sigma_{l.l.r.}\right)_{compr}} \left(\sigma_{l.l.r.}\right)_{compr} W. \tag{9.74}$$

The equation of the prolonged resistance curve for bending in the coordinates $M_{t_{fr}}$ and t_{fr} is

$$M_{t_{fr}} = \mu_{t_{fr}} \sigma_{scompr} W_{compr} \tag{9.75}$$

The prolonged resistance curve for bending is often expressed in the coordinates $\sigma_{bend} = \frac{M_{t_{fr}}}{W_{compr}}$ and t_{fr}. In this

case, its equation has the form

$$\left(\sigma_{bend}\right)_{t_{fr}} = \mu_{t_{fr}} \sigma_{compr} \qquad (9.76)$$

Experimental investigations on prolonged strength in the case of pure bending have been conducted on wood, DSP-plastics and plastic glasses. The method of tests and processing of the experimental data is analogous to those for tests on compression and tension. The limit of ultimate strength for bending was determined preliminarily from the results of tests carried out at different loading rates. The form and dimensions of the specimens were described earlier.

It was shown by experiments that with an increase in the loading rate, the value of the maximum (limiting) breaking bending moment increases according to an asymptotic law. As an example, the curve showing the influence of the loading rate on the value of conditional ultimate strength of DSP-B plastic in the coordinates σ_{bend} and V_σ is shown in Fig. 57 for the case of bending.

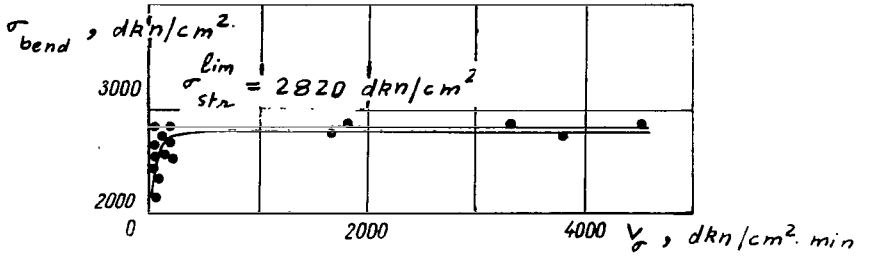


Fig. 57. Influence of the rate of increase in stresses on the value of conditional ultimate strength of DSP-B plastic subjected to bending in the plane parallel to the ply layers.

The values of the limits of ultimate strength on bending, determined from the values of limiting bending moments on the asymptotic branch depending on the loading rate, are given in Table 15. The limits of ultimate strength on bending have been obtained from the formula $\sigma_{bend} = \frac{M}{W}$. The values of the limits of ultimate strength on bending, obtained theoretically from the data of the tests on tension and compression, are also presented here. It is seen that the theoretical and experimental values of the limits of ultimate strength on bending are in good agreement.

Table 15

Material	Limit of ultimate strength on bending σ_{bend}^{lim}, dkn/cm^2	
	Experimental	Theoretical
Pine	875	980
Beech	1122	1250
DSP-B	2820	2680
KAST-V (along warp)	1290	1160
KAST-V (along weft)	1160	1220

The constant bending moments were obtained at a value less than the limiting one, which corresponds to the limit of ultimate strength on bending. As a result, prolonged resistance curves were obtained (Fig. 58), from which the limits of long-term resistance to bending were obtained (Table 16). It is seen from Table 16 that in this case also, the experimental and theoretical values of the limits of long-term resistance to bending are in good agreement.

Table 16

Material	Limit of long-term strength on bending $\sigma_{l.l.r.}^{bend}$, dkn/cm^2	
	Experimental	Theoretical
Pine	520	525
Beech	569	586
DSP-B	1630	1600
DSP-V	750	
DSP-G	803	810
KAST-V (along warp)	1070	1060

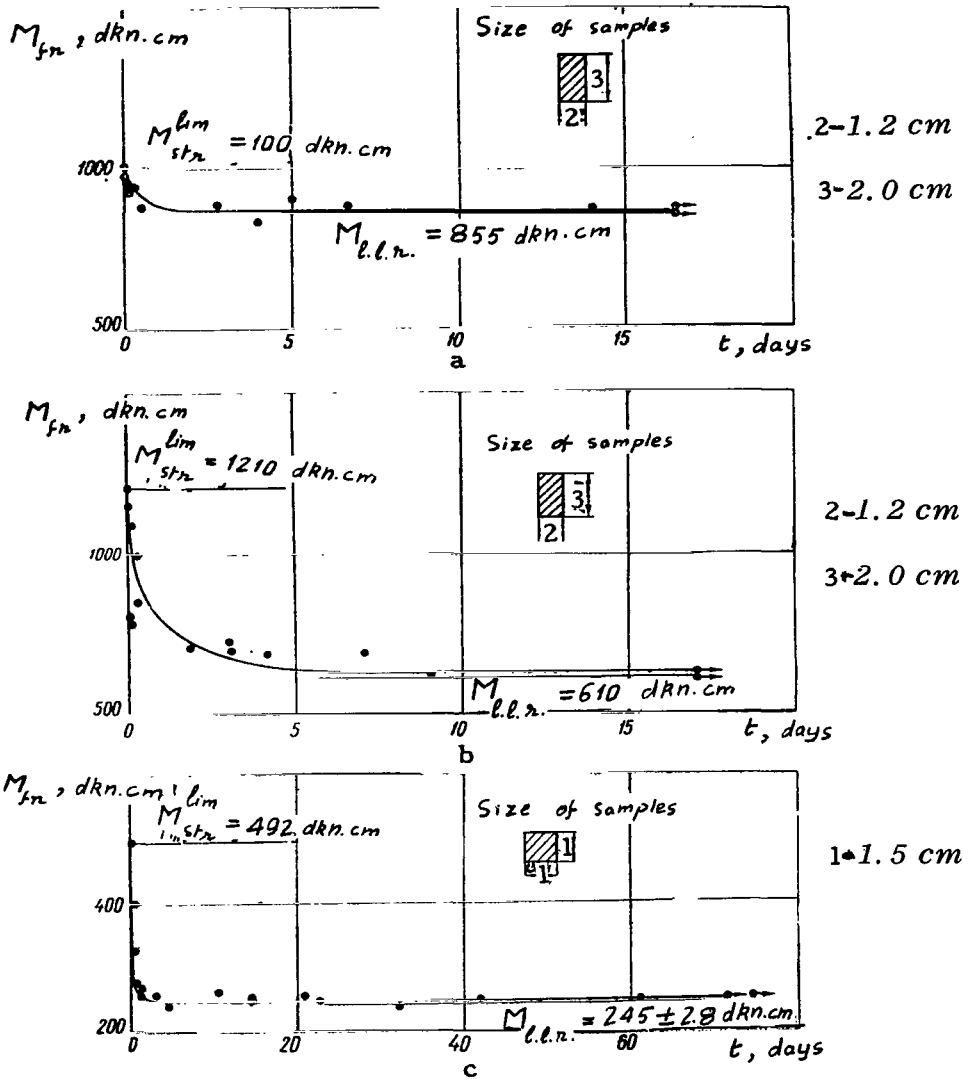


Fig. 58. Long-term resistance curves on bending.
a - KAST-V plastic glass, along warp; b - DSP-V plastic, parallel to the ply layers; c - pine wood, along the fibers.

CARRYING CAPACITY AND DEFORMABILITY ON BENDING IN THE CASE OF DEVELOPMENT OF PLASTIC DEFORMATIONS IN THE STRETCHED ZONE OF THE SECTION

If the material in the section of the bar is situated mainly in the compressed zone, plastic deformations may arise only in

the stretched zone of the section, while the material in the compressed zone works elastically. In this case, the process of deformation with time breaks up into four periods.

In the first period of deformation, instantaneous elastic deformations are formed. Normal stresses along the height of the section are distributed according to the linear law, and the relation between stresses and deformations obeys the law of proportionality. The diagram of normal stresses and the corresponding diagram of relative deformations are shown in Fig. 59, a. Sags and angles of deflection of the sections in the first period are determined by ordinary methods.

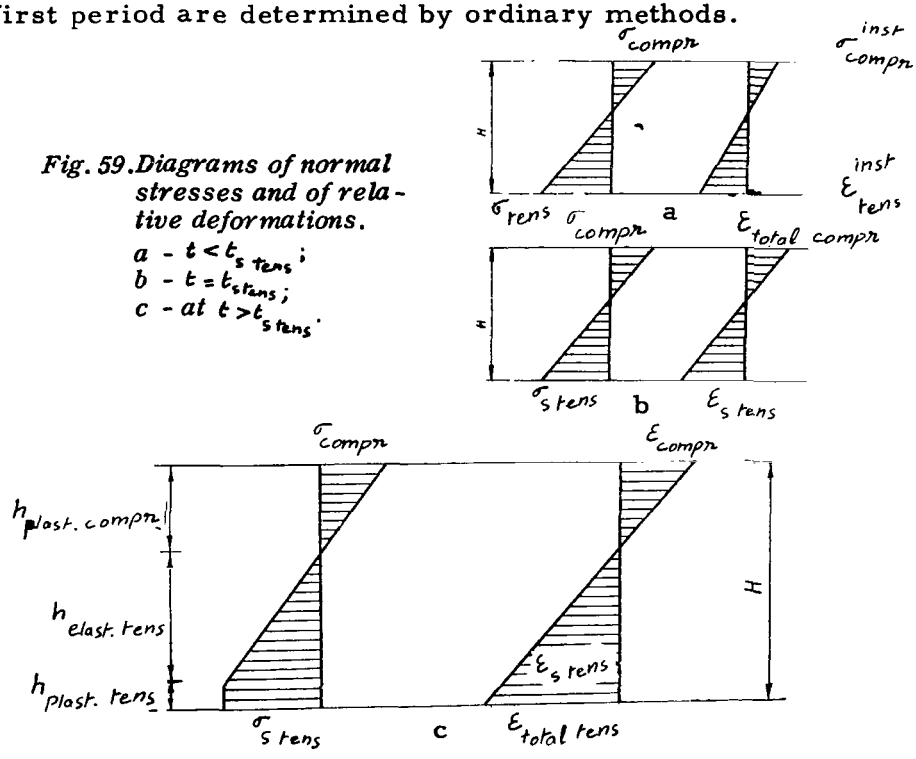


Fig. 59. Diagrams of normal stresses and of relative deformations.
a - $t < t_{s\ tens}$;
b - $t = t_{s\ tens}$;
c - at $t > t_{s\ tens}$.

During the second period of deformation, the elasto-viscous deformations, also obeying the law of proportionality, develop with time in all the fibers of the bar and the stresses along the height of the section change according to a linear law. Angles of deflection of the sections and sags are determined as in the first case, but with the modulus of elasticity changing with time. This period ends after the attainment of the limiting values by the stresses in the stretched end fiber and by elasto-viscous deformations in the course of the final time interval $t_{s\ tens}$. The diagram of normal stresses and the

diagram of relative deformations of the fibers at the end of the time interval $t = t_{s\,tens}$ are shown in Fig. 59, b.

In the third period of deformation, at time interval $t > t_{s\,tens}$, the development of elastic deformations in the neighboring less stressed fibers comes to an end, and the formation of plastic deformations begins. Since most of the fibers of the bar work elastically, the development of plastic deformations in the stretched zone is constrained, which according to the hypothesis of planar section leads to the redistribution of normal stresses along the section on the one hand, and to a more intensive growth of elasto-viscous deformations on the other hand. Constraint of the development of plastic deformations with time leads to the result, that their rate of development does not remain constant and equal to the limiting value, but changes continuously. Hence, at any time interval $t > t_{s\,tens}$, the stresses in the stretched fibers change, decreasing from $\sigma_{s\,tens}$ in the end fiber to $\sigma'_{s\,tens}$ at the boundary of the elastic and the plastic core.

At any time interval $t > t_{s\,tens}$, the diagram of normal stresses, which change with time, has the form shown in Fig. 59, c. Total deformations in the third period grow with time according to a curvilinear law in contrast to the case of bending with development of plastic deformations in the compressed and stretched zones of the section. This may be explained by the influence of the majority of the elastically working fibers, which retards the development of plastic deformations in the stretched fibers.

In the fourth period of deformation, the process of progressing fracture with time takes place. The elastic limit for tension at any time interval $t > t_{s\,tens}$ may be obtained from the formula

$$\sigma_{s\,tens} = \frac{\sigma_{l.n.\,tens}}{1-(1-R_{2\,tens})\,e^{-\alpha_{tens}\left(\frac{t}{t_{1\,tens}}\right)^{c_{tens}}}}. \qquad (9.77)$$

Dimensions of elastic and plastic cores of the section may be determined by using the formula obtained for the case of bending, when plastic deformations develop only in the compressed zone.

Let us denote the ratio of stresses in the compressed and stretched end fibers at time $t > t_{s\,tens}$ by

$$\bar{\eta}_t = \frac{\sigma_{s\,tens}}{\sigma_{compr}}. \tag{9.78}$$

The bending moment in any section of the bar with plastic deformations in the stretched zone is

$$M_x = \bar{\mu}_t \sigma_{s\,tens} W_{tens}, \tag{9.79}$$

where $\bar{\mu}_t$ is the coefficient, which takes into account the increase in carrying capacity at time $t > t_{s\,tens}$ on bending, provided plastic deformations have developed in the stretched part of the section as compared to the limiting elastic carrying capacity of the beam at this very instant.

In order to obtain $\bar{\mu}_t$, the formula obtained in Chapter VIII may be used, in which geometrical properties of the section should be substituted.

The magnitude of the stresses in the compressed end fiber is obtained from formula (9.79), by putting the value of the coefficient μ_t in it:

$$\sigma_{compr} = f(M_x, \sigma_{s\,tens}). \tag{9.80}$$

Using the relation (9.80), the coefficient $\bar{\eta}_t$ may be obtained as a function of bending moment M_x and stresses $\sigma_{s\,tens}$.

$$\bar{\eta}_t = f(M_x, \sigma_{s\,tens}). \tag{9.81}$$

The expression (9.81) is essential for determining the deformations in the elasto-visco-plastic region of work of the material. At the time of determination of deformations in this case of bending, it is essential to have two differential equations of the bent axis of the bar. The first equation is correct for the part of the bar within the limits of which the material acts elastically in the compressed and stretched regions:

$$y'' = \frac{M_x}{E_t I}.$$

The limits of applicability of this equation is determined from the condition

$$M_x = \sigma_{s\,tens} W_{tens}$$

The second equation is correct for the part with elasto-visco-plastic work of material

$$y'' = \frac{\varepsilon_{compr} + \varepsilon_{total\ tens}}{h} . \qquad (9.82)$$

Relative deformation of the compressed end fiber is

$$\varepsilon_{compr} = \frac{\sigma_{compr}}{E_t} . \qquad (9.83)$$

According to Fig. 53

$$\varepsilon_{total\ tens} = \frac{h_{plast.\ tens} + h_{elast.\ tens}}{h_{elast.\ compr}} \varepsilon_{compr} . \qquad (9.84)$$

Putting the value of ε_{compr} and $\varepsilon_{total\ tens}$ in the expression (9.82), we get

$$y'' = \frac{\sigma_{compr}}{E_t h_{elast.\ compr}} \qquad (9.85)$$

or

$$y'' = \frac{\sigma_{s\ tens}}{E_t h_{elast.\ tens}} . \qquad (9.86)$$

If the differential equations are known, sags and angles of deflection of the section are obtained by methods given earlier.

The limiting carrying capacity, limiting deformability and the time till fracture may be determined from the following equality at the time of fracture:

$$\varepsilon_{total\ tens} = \varepsilon_{fr.\ tens} \qquad (9.87)$$

Total deformation of the stretched end fiber grows with time and the law of its growth (Fig. 59, c) may be obtained from the formula:

$$\varepsilon_{total\ tens} = \varepsilon_{s\ tens} \frac{h - h_{elast.\ compr}}{h_{elast.\ tens}} , \qquad (9.88)$$

in which the values of $\varepsilon_{s\ tens}$, $h_{elast.\ compr}$ and $h_{elast.\ tens}$ should be put as functions of the time of action of a constant load.

The law of change of breaking deformation has been obtained earlier for the case of tension.

Solving simultaneously the equations of the change in $\varepsilon_{total\ tens}$ and $\varepsilon_{fr.\ tens}$ with time it is possible to find out the time till fracture and to determine the limiting deformability and carrying capacity by using the formulas, obtained earlier, in which $t = t_{fr.\ tens}$ must be put.

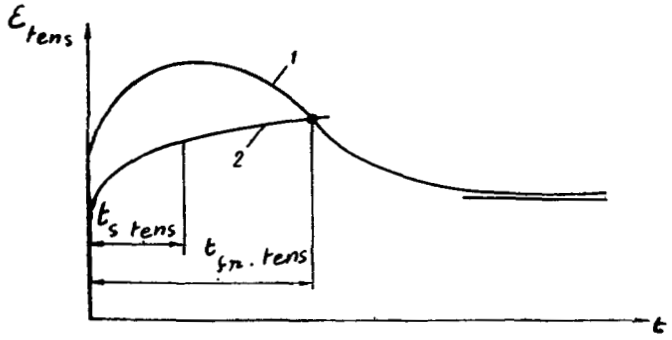


Fig. 60. Determination of time till fracture by the graphical method.

The time till fracture may also be obtained by the graphical method. For this, let us plot the curves showing the relation between $\varepsilon_{fr.\ tens}$ and $t_{fr.\ tens}$ (curve 1) and the relation between $\varepsilon_{total\ tens}$ and t (curve 2) in the coordinates ε and t_{fr} (Fig. 60). The point of intersection of these curves will determine the time till fracture $t_{fr.\ tens}$.

CHAPTER X

BENDING AT A CONSTANT LOADING RATE

DEFORMATION DIAGRAM OF BENDING

In stepped loading of forces, causing constant increase in the value of the bending moment M by holding at every step for time interval T, the laws of change of deformations and resistibility of the bars with time may be obtained from the deformability mechanism considered earlier for linear stressed state.

In the case of bending, total deformations in the elastic region of work of the material in stepped loading consist of the sum of the instantaneous and elasto-viscous deformations with the consideration of heredity at every step due to the influence of loadings in the preceding steps.

After attaining the limiting values of stresses in the compressed or stretched fibers in stepped loading, further development of deformations is determined by the elastic, viscous and plastic properties of the material.

On bending in the elastic region of work of the material, let us assume that the hypothesis of planar sections and the law of proportionality between the stresses and deformations with time are valid. The increase in deformations at every step may be obtained from the value of change in the curvature of the bar on stepped loading.

Let us consider the deformation process of bending in the elastic region of work of the material in tests with a constant loading rate, assuming that the load increases with time according to the linear law:

$$P = v_{incr}\, t ,$$

where t is the time of testing.

Depending on the scheme of loading the bending moment is

$$M = P\gamma ,$$

where γ is the coefficient characterizing the scheme of bending of the bar.

The rate of increase of the bending moment is

$$v_M = \frac{M}{t} = \frac{P\gamma}{t}.$$

Considering the increase in curvature due to the increase in the bending moment, we get

$$\Delta\left(\frac{1}{\rho_{total}}\right) = \frac{\Delta M_x}{IE_t} = \frac{\Delta M_x}{IE_{prol}}\left[1 - ke^{-\alpha\left(\frac{t}{t_1}\right)^c}\right]. \qquad (10.1)$$

Integrating the equation (10.1) with respect to the infinitely small steps of loading dM_x and infinitely small deformations $d\left(\frac{1}{\rho_{total}}\right)$, we get

$$\frac{IE_{prol}}{\rho_{total}} = M_x + M_x k e^{-\alpha\left(\frac{t}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} t^{1-nc} \qquad (10.2)$$

The approximate differential equation of the bent axis, in this case, is

$$E_{prol} I y'' = M_x + M_x k e^{-\alpha\left(\frac{t}{t_1}\right)^c} \sum_{n=1}^{n=\infty} \frac{[1-(n-1)c]!}{\left(\frac{\alpha c}{t_1^c}\right)^n} t^{1-nc}. \qquad (10.3)$$

Integrating the equation (10.3) with respect to x, we get the law of change of sags and angles of deflection of the bar.

The equation (10.3) is true for the elastic region of work of the material. The limit of its application may be determined on the basis of the fact that the limiting elastic bending moment is attained when the formation of plastic deformations begins in the stretched end fiber.

The determination of the limiting elastic bending moment is shown in Fig. 61, where curves (1) showing the dependence of the bending moment as a function of time, and (2) showing the change in the limiting bending moment with time, obtained from the values of the elastic limit for the compressed end fiber are shown in terms of the coordinates "bending moment - time of test". The point of intersection of these curves determines the value of the limiting elastic bending moment and the time of test with a constant rate of loading, during which this bending moment will be reached.

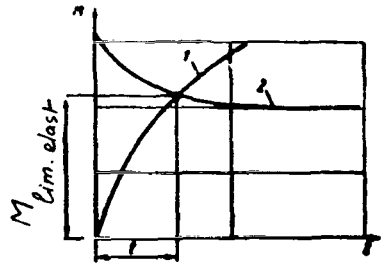


Fig. 61. Determination of the limiting elastic bending moment by the graphical method.

The boundary, which determines the applicability of the equation (10.3), may also be obtained analytically. For this purpose, it is essential to solve simultaneously the equation of the deformation diagram of bending and the equation showing the relation between limiting elastic bending moment and time.

On loading at a constant rate, the diagram of total deformations in the case of bending in the elasto-plastic region of work of the material may be obtained by using the differential equation of the bent axis, in which the influence of the development of plastic deformations is taken into consideration.

Considering the increase in the curvature due to the increase in the bending moment and the time of test, we get

$$\Delta\left(\frac{1}{\rho_{total}}\right) = \frac{\sigma_{s\,compr}}{h_{elast.\,compr}\,E_t} - \frac{\sigma'_{s\,compr}}{h'_{elast.\,compr}\,E'}, \qquad (10.4)$$

where $\sigma_{s\,compr}$, $h_{elast.\,compr}$, and E_t are respectively the elastic limit for compression, height of the compressed part of the elastic core of the section and the modulus of elasticity which changes with time under the action of the bending moment $M_x + \Delta M_x$ in the course of the time interval $t + \Delta t$; $\sigma'_{s\,compr}$, $h'_{elast.\,compr}$, E'_t are the same quantities under the action of the bending moment M_x in the course of time t.

The appearance of the plastic deformations is related to the redistribution of the normal stresses along the height of the section of the bar, which depends on the value of the bending moment and the time of its action. Hence, the heredity resulting from the influence of the preceding loads, while loading, is duly accounted for by the change in the diagrams of normal stresses and the corresponding diagram of relative fibrous

deformations, and consequently, by the curvature of the bar at the particular time of loading.

The increase in the bending moment in the process of loading causes a change in the normal stresses in the plastic and the elastic cores of the section and in their dimensions, which is connected with the change in the diagram of relative fibrous deformations.

Consequently, the relation between curvature and bending moment at the given rate of loading is determined at each time interval by the quantities M_x and $\frac{1}{\rho_{total}}$ with due consideration of the redistribution of stresses and the development of elastic, viscous and plastic deformations with time.

Thus, the differential equation of the bent axis of the bar in the case of appearance of plastic deformations in the compressed zone of the section in tests with a constant loading rate has the same form as in the case of prolonged action of a constant load, but in contrast to this, the bending moment changes with time depending upon the rate of loading, i. e.,

$$y'' = \frac{\sigma_{s\,compr}}{h_{elast.\,compr} E_t}, \qquad (10.5)$$

where $\sigma_{s\,compr} = f(v_{loading})$; $h_{elast.\,compr} = f(M_x, v_{loading})$; $E_t = f(v_{loading})$; $v_{loading} = f(M_x, t)$; and t is the time of test.

The equation (10.5) is applicable when the bending moment changes in the interval from the limiting-elastic moment $M_{lim.\,elast}$ to the limiting elasto-viscous moment $M_{lim.\,vis}$ which, when exceeded, leads to the appearance of plastic deformations in the stretched zone of the bar.

The limiting elastic moment may be obtained from the formula

$$M_{lim.\,elast} = \sigma_{s\,compr} W,$$

where $\sigma_{s\,compr}$ is the elastic limit of the extreme compressed end fiber. For this purpose, it is essential to solve

simultaneously the equations of the change in bending moment on loading in the elastic region and the equation of change in limiting elastic moment with time.

Fig. 62 shows the method for determining the limiting elastic moment, which corresponds to the point of intersection of the curve 1 showing the change in limiting elastic moment with time and the curve 2 showing the change in bending moment on loading at a constant rate. The curve is plotted in the coordinates "bending moment -- time of test".

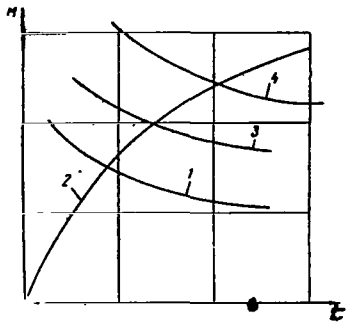


Fig. 62. Determination of the limiting elastic and the limiting elasto-viscous bending moments by the graphical method.

In order to determine the limiting elasto-viscous bending moment, it is essential to solve simultaneously the equation of change in bending moment on loading, when plastic deformations have developed in the compressed zone of the bar, and the equation of change in limiting elasto-viscous bending moment with time. The latter is determined from the formula:

$$M_{lim.\,vis} = \mu_{lim}\,\sigma_{s\,compr}\,W,$$

where $\mu_{lim} = f(\eta_{lim})$; $\sigma_{s\,compr} = f(t)$; $\eta_{lim} = f(\sigma_{s\,compr}; \sigma_{s\,tens})$; $\sigma_{s\,tens} = f(t)$.

The value of $M_{lim.\,vis}$ (see Fig. 62) may be obtained from the point of intersection of curve 2 and the curve 3 showing the change in limiting elasto-viscous bending moment with time.

In order to obtain the equation of deformation diagram for the case of bending at a constant loading rate, when plastic deformations develop in the compressed and stretched zones of the section, the following differential equation of the bent axis may be used:

$$y'' = \frac{\sigma_{s\,compr}}{h_{elast.\,compr}\,E_t}, \qquad (10.6)$$

where the quantities $\sigma_{s\,compr}$, $h'_{elast.\,compr}$ and E_t should be expressed as a function of the time of test, which will give the relation of bending deformations as a function of the bending moment increasing at a constant rate. The equation (10.6) is valid in the interval of bending moments from limiting elasto-viscous moment to the limiting bending moment at fracture.

The limiting elasto-viscous bending moment may be obtained by the method mentioned earlier, and the limiting bending moment at fracture may be obtained by the formulas for determining the limiting carrying capacity under the action of a sustained constant load, in which we must substitute the characteristics depending on the time till fracture. The value of the limiting bending moment is determined by solving simultaneously the equation of the relation between the limiting bending moment and time and the equation of the change in bending moment with time, provided the plastic deformations have developed in the compressed and stretched zones of the bar. The value of the limiting bending moment (see Fig. 62) is determined from the point of intersection of the curve 2 showing the growth of bending moment during test and the curve 4 of change in limiting bending moment with time.

The differential equations of the bent axis are integrated by the same methods as in the case of a sustained action of a constant load.

EXPERIMENTAL INVESTIGATIONS ON STRENGTH AND DEFORMABILITY OF BEAMS OF A RECTANGULAR SECTION ON PURE BENDING

Tests on pure bending of DSP-plastics have been conducted at a temperature $t = 20^{\circ}C$ with a constant loading rate $V_{tens} = 24$ dkn/min on planes parallel and perpendicular to the ply layers.

During the test, the sag was determined at the middle point of the sample on a base $l = 75$ mm in its working part.

The diagrams of deformations in the case of pure bending were plotted in the coordinates along the ordinate of which the fictitious normal stresses in the farthest fiber, determined from the formula $\sigma = \frac{M}{W}$ were plotted, and the relative fictitious

deformations of the farthest fiber were plotted along the abscissa. The fictitious relative deformation was determined from the formula

$$\varepsilon = \frac{4hf}{l^2},$$

since

$$f = \frac{Ml^2}{8EI} = \frac{\sigma W l^2}{8EI} = \frac{\varepsilon l^2}{4h},$$

where f is the sag at the middle point of the selected base in the working part of the specimen, h is the height of the cross section of the specimen in the working part and l is the length of the base.

By using the theory of bending of beams of elasto-visco-plastic material, it is possible to pass on from the fictitious stresses and fictitious relative deformations to the real stres-

Table 17

Mechanical characteristics on bending		DSP-B		DSP-V		DSP-G	
		Parallel to the ply layers	Perpendicular to the ply layers	Parallel to the ply layers	Perpendicular to the ply layers	Parallel to the ply layers	Perpendicular to the ply layers
Ultimate, strength σ_{str}, dkn/cm^2	Average	2670	2830	1660	1190	1450	1550
	Maximum	2840	2900	1810	1270	1510	1615
	Minimum	2520	2730	1615	1030	1380	1480
Conditional limit of proportionality $\sigma_{l.p.}$, dkn/cm^2	Average	2070	1945	1200	1050	1260	800
	Maximum	2200	2180	1290	1230	1310	870
	Minimum	2000	1800	1130	970	1215	730
Modulus of elasticity from machine tests of short duration $E_{mach} \times 10^5$ dkn/cm^2	Average	3.5	2.9	1.39	2.03	1.66	1.60
	Maximum	3.7	3.1	1.42	2.26	1.71	1.63
	Minimum	3.3	2.6	1.33	1.97	1.62	1.58
Instantaneous modulus of elasticity $E_{inst} \times 10^5$, dkn/cm^2	Average	3.9	3.3	1.75	2.07	2.00	1.70
	Maximum	4.0	3.5	1.79	2.11	2.06	1.75
	Minimum	3.6	2.9	1.68	2.01	1.95	1.68
Relative fibrous deformation at fracture, $\varepsilon_{fr} \times 10^2$	Average	1.07	1.60	1.22	0.63	1.00	1.80
	Maximum	1.25	2.13	1.78	0.98	1.27	2.01
	Minimum	0.97	1.20	1.07	0.43	0.84	1.59

ses in the stretched or compressed end fibers, and to the real relative deformation of these fibers.

Table 17 shows the average results of 20 tests on DSP-plastic subjected to pure bending in the planes parallel and perpendicular to the ply layers.

Tests on pure bending of KAST-V plastic glass were conducted at t = 20°C and humidity in the delivery condition with a constant loading rate V_{tens} = 240 dkn/min. KAST-V plastic glass was bent in the plane of the warp or weft, parallel and perpendicular to the layers of the glass tissues. Diagrams of total deformations were also recorded during the test, from which mechanical characteristics of the plastic glass were determined. Typical diagrams of total deformations of KAST-V subjected to pure bending are shown in the Fig. 63.

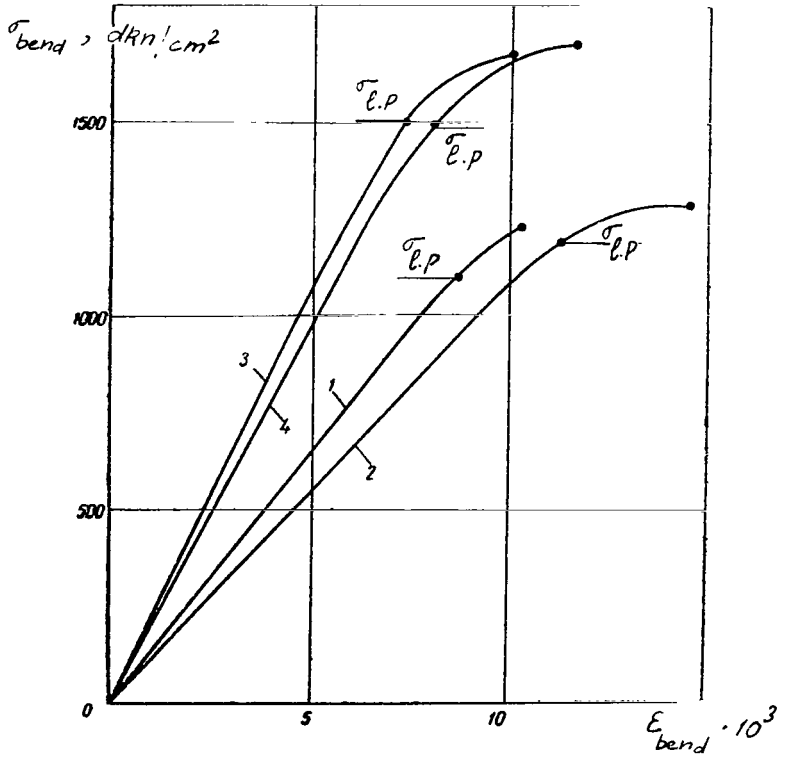


Fig. 63. Deformation diagram of KAST-V subjected to pure bending.
1 - along weft, perpendicular to the plane of pressing; 2 - along weft, parallel to the plane of pressing; 3 - along warp, parallel to the plane of pressing; 4 - along warp, perpendicular to the plane of pressing.

The mechanical properties of KAST-V plastic glass subjected to pure bending are shown in Table 18 as the average of 15 tests in every case.

Table 18

Mechanical properties of KAST-V subjected to bending		Along weft		Along warp	
		Parallel to the plane of pressing	Perpendicular to the plane of pressing	Parallel to the plane of pressing	Perpendicular to the plane of pressing
Ultimate strength σ_{lim}, dkn/cm^2	Average	1225	1210	1625	1580
	Maximum	1290	1250	1670	1710
	Minimum	1130	1150	1570	1510
Conditional limit of proportionality $\sigma_{l.p}$, dkn/cm^2	Average	1100	1000	1315	1290
	Maximum	1190	1100	1500	1480
	Minimum	1050	800	1150	1150
Modulus of elasticity from machine tests of short duration $E_{mach} \cdot 10^5$, dkn/cm^2	Average	1.06	1.29	2.13	1.86
	Maximum	1.10	1.54	2.14	2.22
	Minimum	1.04	1.15	2.13	1.60
Relative fibrous deformation at fracture $\varepsilon_{fr} \cdot 10^2$	Average	1.32	1.05	0.95	0.96
	Maximum	1.46	1.10	0.96	1.17
	Minimum	1.13	0.815	0.94	0.78

Besides the tests on laminated plastics, experiments were conducted on pure bending of bars of rectangular cross section of pine and oak woods. Mechanical properties of each type of wood, as the average of 10 tests, are presented in Table 19.

On the basis of the results of the tests on pure bending, the applicability of the theory of limiting carrying capacity was verified in order to determine the conditional ultimate strength

Table 19

Mechanical characteristics on bending		Pine	Oak
Ultimate strength σ_{str}, dkn/cm^2	Average	1186	1263
	Maximum	1495	1445
	Minimum	762	1030
Conditional limit of proportionality $\sigma_{l.p}$, dkn/cm^2	Average	920	930
	Maximum	1120	1135
	Minimum	510	840
Modulus of elasticity from machine tests of short duration $E_{mach} \cdot 10^5$, dkn/cm^2	Average	1.07	1.05
	Maximum	1.41	1.13
	Minimum	0.84	0.97
Relative fibrous deformation at fracture $\varepsilon_{fr} \cdot 10^2$	Average	1.40	1.25
	Maximum	1.65	1.50
	Minimum	0.90	1.10

of bending by using ultimate strength of tension and compression. It is known that, according to the theory of limiting carrying capacity, in the case of short-duration tests, the trigonal-trapezoidal diagram of normal stresses is utilized and the development of plastic deformations in the stretched fibers of the bar is neglected. This allowance is permissible, since, at comparatively short test time, the value of plastic deformations in the stretched zone of the bar at the time of fracture is insignificant and may be neglected. Then the value of the conditional ultimate strength of bending is

$$\sigma_{bend} = \mu \sigma_{compr},$$

whereas for a bar with rectangular cross section, the coefficient

$$\mu = \frac{3\eta - 1}{\eta + 1},$$

and the coefficient η is equal to the ratio of ultimate strengths of compression and tension, which are obtained at the same rates of increase of stresses, as in the case of bending.

A comparison of the experimental and the theoretical values of the conditional ultimate strength of bending of laminated plastics is given in Table 20.

Table 20

Material	Direction of fracture	Conditional limit of proportionality at fracture σ_{bend}, dkn/cm^2	
		Experimental	Theoretical
DSP-B	Parallel to the ply layers	2670	2680
	Perpendicular to the ply layers	2830	2680
DSP-V	Parallel to the ply layers	1660	1380
	Perpendicular to the ply layers	1190	1380
DSP-G	Parallel to the ply layers	1450	1490
	Perpendicular to the ply layers	1550	1490
KAST-V	Along warp	1225	1100
	Along weft	1025	1115

It is seen that a completely satisfactory concurrence of the experimental and theoretical values has been obtained, which permits to recommend the formula $\sigma_{bend} = \mu \sigma_{compr}$ for short-duration tests. It should also be mentioned that this formula also shows good agreement between the theoretical values of conditional ultimate strength of bending and the experimental values in the case of other types of cross section of the bar.

CONCLUSION

In this work, resistibility, strength and deformability of laminated plastics and wood have been studied experimentally and theoretically for the cases of compression, tension and bending under loads which are constant or variable with time. In order to clarify the influence of the time factor on the mechanical characteristics of the material, prolonged tests under the action of constant loads are of maximum interest, because in this case, the laws of deformation and change in strength depend only on the properties of the material and do not become complicated because of change in load during the test. It has been established that the presence of instantaneous elastic, elasto-viscous and plastic deformations is characteristic of the materials under consideration. An analysis of the deformation diagrams showed that instantaneous elastic deformations follow Hooke's law right up to the fracture of the material. Elasto-viscous deformations, developing with time according to exponential law, also obey the law of proportionality between stresses and deformations at each time interval of action of the constant load. However, plastic deformations under the action of a constant load increase with time at a constant rate, the value of which does not have a linear relation with stress. Ultimate strength and elastic limit decrease simultaneously with a change in deformations of the material with time. Considering both the phenomena, it may be established, that at constant temperature the principal laws guiding strength, resistibility and deformability under the axial action of a load, characterizing elasto-visco-plastic work of material are as follows:

(a) law of development of deformations with time in the elastic region of work of material -- creep curve at a constant load;

(b) law of change in ultimate strength with time -- curve of prolonged resistance at constant loads;

(c) relation between the rate of development of plastic deformation and the value of elastic limit or the relation between elastic limit and the limiting elastic deformation.

Using the analytical expressions for these rules, which are best satisfied by experimental data, it is possible to determine, with sufficient accuracy, the behavior of the material at constant given loads or deformations, as well as at loads and deformations changing according to a particular law, etc. The essential characteristics of strength and deformability of the material, which are included in the analytical relation, are determined by the methods given in this work.

Taking the rules of the variation of strength and deformability, with time, for tension and compression as the basis, it is possible to have analytical solutions of the problems of bending in the elastic and elasto-plastic regions of work of material, and also in the limiting state. Here, it is essential to consider the difference in absolute value of the characteristics of strength and resistibility to compression and tension.

Strength, resistibility and deformability of laminated plastics and wood at normal temperature have been considered; experimental works have also been conducted to study the influence of increased temperature on the mechanical characteristics of laminated plastics [13, 14].

Experiments show that the creep curves and prolonged resistance curves at increased temperatures have the same characteristics as at normal temperature. Thus, analytical expressions of the rules of the change in strength, resistibility and deformability with time at increased temperature are the same as those at normal temperature.

After obtaining the boundaries of ultimate strength, prolonged strength limit, instantaneous and prolonged moduli of elasticity and other constants as a function of temperature, it is possible to put them in the formulas used for the work of material at normal temperatures. The functional relation of the characteristics of instantaneous and prolonged strength and deformability with temperature, in the temperature range up to 200°C, is expressed most accurately by exponential functions.

The work undertaken makes it possible to give a more basic approach to the estimation of the mechanical properties of the given materials.

The principal laws for strength, resistibility and deformability may be used in examining other types of stressed states, although more complicated problems of the stressed and deformed states of material require the introduction of additional hypotheses and their experimental verification.

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